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Research Article

Approximate Analytical Solution to the Problem on the Theoretical Profile of Dimensionless Velocity According to the Thickness of the Boundary Layer in Turbulent Boundary Layer Flow Based on the Solution of the Abele Second Differential Equation Genus by the Method of Successive Approximations with Additional Assumptions

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Abstract

In the article an exact analytical solution of the differential equation for shear stresses in a turbulent boundary layer, which is a special case of the so-called Abel differential equation of the second kind, was found, obtained using a special Lambert function, while previously it was believed that it is not solvable in quadratures. In addition, several other important solved special cases of this equation were obtained. The analytical solutions obtained in the article mainly differ from the previously available either numerical or approximate solutions of the problem. The obtained solution in dimensionless form is a theoretical profile of the dimensionless velocity along the thickness of the boundary layer in turbulent flow in the boundary layer. An approximate solution of this equation by the method of successive approximations with additional assumptions, as well as the corresponding analytical approximation dependence, were also obtained.

Keywords: Theoretical; Modeling; Mathematical; Velocity; Coordinate; Dimensionless; Profile; Heat Exchange; Turbulent; Flow; Boundary Layer; Abel Differential Equation; Of the Second Kind; Of the First Kind; Lambert Function; Approximate; Method of Successive Approximations

Introduction

Solution of the problem of velocity profile in a plane turbulent flow of incompressible coolantcan be given both from theoretical considerations and by introducing semi-empirical and empirical dependencies.

It should be noted that a number of authors, for example, Deissler, Van Driest, Reithard, Levich, Lin, Loitsyansky and others, have proposed empirical and semi-empirical relationships for determining the velocity profile in a turbulent boundary layer [5,6]. These profiles in the region of joint action of molecular and turbulent viscosity have a rather complex configuration, piecewise continuity in the form of a significant number of individual sections. For practical purposes, Karman proposed dividing the boundary layer into three zones and approximating two of them with logarithmic formulas; it is also possible to replace the universal law of velocity distribution in a turbulent flow with a power law, where the logarithmic velocity profile is the envelope of a family of power profiles.

The logarithmic velocity profile itself can be considered as a certain fact of the existence of a universal law of distribution of dimensionless velocity in a turbulent boundary layer during flow around the vicinity of an impermeable plate by a turbulent unlimited isothermal flow of an incompressible coolant [5].

This study aims to obtain a theoretical solution for the velocity profile in a flat turbulent boundary layer based on the solution of a differential equation for shear stresses.

The rationale for the calculation model of turbulent wall flow near the surface, both physical and mathematical, and the description of the turbulence model for these conditions was made in the monograph [1], therefore, within the framework of this work, we will only present the resulting expressions. General shear stressτis determined as the following sum [1]:

$$\tau=\mu\frac{dw}{dy}+\rho\psi\zeta\left(l\frac{dw}{dy}+\frac{1}{2}\frac{d^2w}{dy^2}l^2\right)^2 \quad (1)$$

Where μ— dynamic viscosity; w — longitudinal velocity; y — transverse coordinate;ρ— density; l — linear dimension such as the length of the mixing path;ζ— replacement coefficient;ψ— correlation coefficient.

Turbulence constantκwill be determined by the following expression:

$$y\cdot\kappa=l\cdot\sqrt{\psi\zeta} \quad (2)$$

Substituting (2) in (1) and carrying out elementary transformations, we obtain:

$$\tau=\mu\frac{dw}{dy}+\rho\kappa^2y^2\left(\frac{dw}{dy}\right)^2+\rho\frac{\kappa^3y^3}{\sqrt{\psi\zeta}}\frac{dw}{dy}\frac{d^2w}{dy^2}+\frac{1}{4}\rho\frac{\kappa^4y^4}{\psi\zeta}\left(\frac{d^2w}{dy^2}\right)^2 \quad (3)$$

IN [1] it is rightly pointed out that in the boundary layer there is a certain region in which the interaction of large-scale and small-scale pulsations takes place, significantly influencing the shear stress, therefore the main contribution to it will be made by the first and third terms of the right-hand side of (3):

$$\tau=\mu\frac{dw}{dy}+\rho\frac{\kappa^3y^3}{\sqrt{\psi\zeta}}\frac{dw}{dy}\frac{d^2w}{dy^2} \quad (4)$$

Let us reduce equation (4) to dimensionless form by introducing dimensionless coordinates $\eta=yw_*/\nu$ and speed $\varphi=w/w_*$ ($\nu=\mu/\rho$— kinematic viscosity; $w_*=\sqrt{\tau_w/\rho}$ — dynamic speed or "friction speed"):

$$\frac{d\varphi}{d\eta}+\frac{\kappa^3}{\sqrt{\zeta\psi}}\eta^3\frac{d\varphi}{d\eta}\frac{d^2\varphi}{d\eta^2}=1 \quad (5)$$

In progress [1] it is indicated that (5) is a special case of the so-called Abel differential equation of the second kind, which in this version does not lead to quadratures, as indicated in the same work [1]. We will show further that without the use of special functions this is so, however, the use of the special Lambert function [3] allows us to solve this equation in quadratures, and in some cases to obtain its exact analytical solution.

Therefore, we have a differential equation of the form:

$$\frac{dy}{dx}+f(x)\frac{dy}{dx}\frac{d^2y}{dx^2}=1 \quad (6)$$

Abel differential equation of the second kind, according to [4], has the following form (according to classification [4] - 4.11(b)):

$$[g_1(x)y+g_0(x)]\frac{dy}{dx}=f_2(x)y^2+f_1(x)y+f_0(x) \quad (7)$$

If we take in the equation (5) $\frac{d\varphi}{d\eta}=y(x)$, then we get the following equation:

$$y+f(x)y\frac{dy}{dx}=1 \quad (8)$$

The last equation (8) is a special case of (7) in the case g0(x)=0, f2(x)=0, f1(x)=–1, f0(x)=1, namely:

$$g_1(x)y\frac{dy}{dx}=-y+1 \quad (9)$$

In progress [4] it is indicated that quadratures are possible in the following three cases.

First, if:

$$g_0(x)\left(2f_2(x)+\frac{dg_1(x)}{dx}\right)=g_1(x)\left(f_1(x)+\frac{dg_0(x)}{dx}\right)\wedge g_1 \quad (10)$$

Substituting the values of the functions, we obtain:

$$0\cdot\left(2\cdot0+\frac{dg_1(x)}{dx}\right)=g_1(x)(-1+0)\wedge g_1(x)\neq0 \text{ or } 0=-g_1(x)\wedge g_1(x)\neq0 \quad (11)$$

Consequently, there is a contradiction, so in this case the solution of the equation does not lead to quadratures.

Secondly, the solution of equation (8) can be obtained in quadratures by reducing the Abel equation of the second kind of type 4.11(b) [4] to the Abel equation of the second kind of type 4.11(a) [4]. To do this, it is necessary that f0(x)≡0, which contradicts equation (8).

IN-Thirdly, the solution of equation (8) can be obtained in quadratures by reducing the Abel equation of the second kind to the Abel equation of the first kind [4]. Let us consider the Abel equation of the second kind 4.11(c) [4]:

$$[g_1(x)y+g_0(x)]\frac{dy}{dx}=f_3(x)y^3+f_2(x)y^2+f_1(x)y+f_0(x)$$. (12)

From (12) it is clear that equation (8) is a special case of (12) in the case g0(x)=0, f3(x)=0, f2(x)=0, f1(x)=–1, f0(x)=1.

Reduce the equation (12) to the Abel equation of the first kind 4.10 [4]:

$$\frac{dy}{dx}=f_3(x)y^3+f_2(x)y^2+f_1(x)y+f_0(x) \quad (13)$$

It is possible when implementing the substitution

$$g_1(x)y+g_0(x)=\frac{1}{u(x)} \quad , (14)$$

If g1(x)y+g0(x)≠0 and g1(x)≠0.

By the following requirements for equation (8) are satisfied, since g0(x)=0, g1(x)≠0 and g1(x)y≠0.

To solve the Abel equation of the first kind in quadratures, the following conditions must be met [4].

The solution in quadratures for cases 4.10(a), (b), (c) [4] is possible for f3(x)≠0. Quadratures for case 4.10(g) are possible for f0(x)≡0. The transition of the Abel equation of the first kind to the Bernoulli equation is possible at f0(x)≡0 and f2(x)≡0 — case 4.10(d) [4]. Reduction to an equation with separable variables is possible for f0(x)≡0, f1(x)≡0 and $\frac{d}{dx}\left(\frac{f_3(x)}{f_2(x)}\right)=cf_2(x)$ (c is a constant) — case 4.10(e) [4]. For the solution in case 4.10(g) [4] it is necessary that f3(x)≠0.

It is obvious that for equation (8) the above conditions are not met, therefore it cannot be reduced to quadratures by traditional methods, which is rightly indicated in the work [1].

The general solution of this differential equation is(6) can be obtained by introducing the special Lambert function W(x):

$$y(x)=\int W\left(\frac{\exp\left(-\int\frac{dx}{f(x)}-1\right)}{C_1}\right)dx+x+C_2, \quad \text{------(15)}$$

Where $W(x)\cdot\exp(W(x))=x$ — Lambert function; C1 and C2 are constants.

In simpler special cases, integration can be carried outfor solving (15). For example, if f(x)=1, f(x)=x, f(x)= $\sqrt{x}$, then the solution (15) will look like this:

$$y(x)=-\frac{1}{2}W\left(\frac{e^{-x-1}}{C_1}\right)^2-W\left(\frac{e^{-x-1}}{C_1}\right)+x+C_2, \quad (16)$$

$$y(x)=e^{-1}\mathrm{E}\left(1,W\left(\frac{e^{-1}C_1}{x}\right)\right)C_1+W\left(1,W\left(\frac{e^{-1}C_1}{x}\right)\right)x+x+C_2, \quad (17)$$

$$y(x)=-\frac{1}{2}W\left(\frac{e^{-2\sqrt{x}-1}}{C_1}\right)^2\sqrt{x}-\frac{1}{2}W\left(\frac{e^{-2\sqrt{x}-1}}{C_1}\right)^3-W\left(\frac{e^{-2\sqrt{x}-1}}{C_1}\right)\sqrt{x}-\frac{3}{8}W\left(\frac{e^{-2\sqrt{x}-1}}{C_1}\right)^2-$$
$$-\frac{1}{2}W\left(\frac{e^{-2\sqrt{x}-1}}{C_1}\right)+x+C_2, \quad (18)$$

Where Ei(a, z) is the integral exponential function. For Re(z)>0: $\mathrm{E}(a,z)=\int^{\infty} e^{-zt}t^{-a}dt$ This special function is well known; its properties, such as branches, singular points, etc., are given in the specialized mathematical literature [2].

Solution of the equation (8), naturally, it is even easier to obtain:

$$y(x)=W\left(\frac{\exp\left(-\int\frac{dx}{f(x)}-1\right)}{C_1}\right)+1. \quad (19)$$

For equation (19) solutions can be obtained for all elementary functions f(x), as well as for some special functions f(x); as an illustration, we present only solution (19) for the Lambert function, i.e. when f(x)=W(x):

$$y(x)=1+ \quad (20)$$

$$+\exp\left(\mathrm{E}(1,-W(x))-C_1-1-\frac{x}{W(x)}-W\left(\exp\left(\mathrm{E}(1,-W(x))-C_1-1-\frac{x}{W(x)}\right)\right)\right).$$

Values of constants$WITH_1$and C2 in solution (15) for equation (6) we find from the following asymptotic boundary conditions:

atη→0: $\frac{d\varphi}{d\eta}\to\frac{1}{1+(n\eta)^4}\approx 1-(n\eta)^4$, (21)

Where n=0.124;

atη>>0:, (22) $\frac{d\varphi}{d\eta}\to\frac{1}{\varkappa\eta}$

Where =0.4..

Now let's turn to determination of functionsψAndζ, included in equation (5). In work [1], based on the analysis of the equations of turbulent transfer, the following relationship was derived:

$$\frac{1}{\kappa\eta}+\frac{\kappa}{\sqrt{\psi\zeta}}=1. \quad (23)$$

Therefore, we can obtain the dependence: $\sqrt{\psi\zeta}$:

$$\sqrt{\psi\zeta}=\frac{\kappa^2\eta}{\kappa\eta-1}. \quad (24)$$

Thus, the resulting differential equation will look like this:

$$\frac{d\varphi}{d\eta}+\kappa\eta^2(\kappa\eta-1)\frac{d\varphi}{d\eta}\frac{d^2\varphi}{d\eta^2}=1. \quad (25)$$

Solution of the equation (25) we obtain on the basis of solution (15):

$$\varphi(\eta)=\int\frac{\eta}{\kappa\eta-1}\exp\left[-\frac{W\left(\dfrac{\eta e^{-1}}{e^{\frac{C_1}{\kappa}}e^{\frac{1}{\kappa\eta}}(\kappa\eta-1)}\right)\kappa\eta+C_1\eta+\kappa\eta+1}{\kappa\eta}\right]d\eta+\eta+C_2. \quad (26)$$

Now we should determine the constants C1 and C2 from the boundary conditions (21) and (22):

$$\lim_{\eta\to\infty}\left(\frac{d\varphi}{d\eta}\right)=W\left(\frac{e^{-1}}{\kappa e^{\frac{C_1}{\kappa}}}\right)+1. \quad (27)$$

Therefore, if $W\left(\dfrac{e^{-1}}{\kappa e^{\frac{C_1}{\kappa}}}\right)+1=0$, That:

$$C_1=\kappa\ln\left(-\frac{1}{\kappa}\right). \quad (28)$$

Now, after substituting (28) into (26), we get:

$$\lim_{\eta\to 0}\left(\left.\frac{d\varphi}{d\eta}\right|_{C_1=\kappa\ln\left(-\frac{1}{\kappa}\right)}\right)=1, \quad (29)$$

Which coincides with the limit value of formula (21).

We determine the constant C2 from the equality condition (which does not contradict condition (21):

atη→0:φ→0; (30)

$$\lim_{\eta\to 0}\left(\frac{d\varphi}{d\eta}\right)=\left.\int\frac{\eta}{\kappa\eta-1}\exp\left[-\frac{W\left(\dfrac{\eta e^{-1}}{e^{\frac{C_1}{\kappa}}e^{\frac{1}{\kappa\eta}}(\kappa\eta-1)}\right)\kappa\eta+C_1\eta+\kappa\eta+1}{\kappa\eta}\right]d\eta\right|_{\eta=0}+C_2. \quad (31)$$

Hence:

$$C_2=-\left.\int\frac{\eta}{\kappa\eta-1}\exp\left[-\frac{W\left(\dfrac{\eta e^{-1}}{e^{\frac{C_1}{\kappa}}e^{\frac{1}{\kappa\eta}}(\kappa\eta-1)}\right)\kappa\eta+C_1\eta+\kappa\eta+1}{\kappa\eta}\right]d\eta\right|_{\eta=0}. \quad (32)$$

The final general solution of the equation (25) will look like this:

$$\varphi(\eta)=\int\frac{\eta}{\kappa\eta-1}\exp\left[-\frac{W\left(\dfrac{\eta e^{-1}}{e^{\frac{C_1}{\kappa}}e^{\frac{1}{\kappa\eta}}(\kappa\eta-1)}\right)\kappa\eta+C_1\eta+\kappa\eta+1}{\kappa\eta}\right]d\eta+\eta-$$

$$-\left.\int\frac{\eta}{\kappa\eta-1}\exp\left[-\frac{W\left(\dfrac{\eta e^{-1}}{e^{\frac{C_1}{\kappa}}e^{\frac{1}{\kappa\eta}}(\kappa\eta-1)}\right)\kappa\eta+C_1\eta+\kappa\eta+1}{\kappa\eta}\right]d\eta\right|_{\eta=0}= \quad (33)$$

$$=\eta-\int_0^\eta\frac{\eta}{\kappa\eta-1}\exp\left[-\frac{W\left(\dfrac{\eta e^{-1}}{e^{\frac{C_1}{\kappa}}e^{\frac{1}{\kappa\eta}}(\kappa\eta-1)}\right)\kappa\eta+C_1\eta+\kappa\eta+1}{\kappa\eta}\right]d\eta.$$

$$=\eta+\int_0^\eta\frac{\eta}{\kappa\eta-1}\exp\left[-\frac{W\left(\dfrac{\eta e^{-1}}{e^{\frac{C_1}{\kappa}}e^{\frac{1}{\kappa\eta}}(\kappa\eta-1)}\right)\kappa\eta+C_1\eta+\kappa\eta+1}{\kappa\eta}\right]d\eta.$$

After substituting the constant$WITH_1$and simplifications, we get:

$$\varphi(\eta)=\int_0^{\eta} W\left(-\frac{\kappa\eta}{(\kappa\eta-1)e^{\frac{\kappa\eta+1}{\kappa\eta}}}\right)d\eta+\eta. \quad (34)$$

Expression (34) is a theoretical dimensionless velocity profile across the boundary layer thickness, corresponding to equations (5) and (25).

In light of the obtained solution (34), a few words should be said regarding the direct use of the functional asymptotic boundary conditions (21) and (22), as was done in [1] in the approximate solution of equation (5) by the method of successive approximations with additional assumptions.

If we directly substitute the functional boundary condition (22) into the general solution (26), we obtain the following:

$$\lim_{\eta\to\infty}\left(\frac{d\varphi}{d\eta}\right)=W\left(\frac{e^{-1}}{\kappa e^{\frac{C_1}{\kappa}}}\right)+1=\frac{1}{\kappa\eta}, \quad (35)$$

$$C_1=\frac{\ln\left(-\frac{1}{\kappa\eta}\right)\kappa\eta-1}{\eta}. \quad (36)$$

If we substitute$WITH_1$from (36) to (26), then:

$$\frac{d}{d\eta}\int\frac{\eta}{\kappa\eta-1}\exp\left[-\frac{W\left(\frac{\eta e^{-1}}{e^{\frac{C_1}{\kappa}}e^{\frac{1}{\kappa\eta}}(\kappa\eta-1)}\right)\kappa\eta+C_1\eta+\kappa\eta+1}{\kappa\eta}\right]d\eta\equiv-1.$$

If we now use the functional boundary condition (21), we obtain:

$$\lim_{\eta\to 0}\left(\frac{1}{1+(m\eta)^4}\right)= \quad (38)$$

$$=\lim_{\eta\to 0}\left(\frac{d}{d\eta}\left(\int\frac{\eta}{\kappa\eta-1}\exp\left[-\frac{W\left(\frac{\eta e^{-1}}{e^{\frac{C_1}{\kappa}}e^{\frac{1}{\kappa\eta}}(\kappa\eta-1)}\right)\kappa\eta+C_1\eta+\kappa\eta+1}{\kappa\eta}\right]d\eta+\eta+C_2\right)\right).$$

The left side of (38) is equal to one, and the right side is equal to zero, therefore, there is a formal contradiction, which was not the case when using the limiting boundary conditions.

Consequently, the solution indicated in the work [1], obtained by means of the method of successive approximations, using functional asymptotic boundary conditions, is sufficient for practical purposes, as indicated by the comparison of results presented in the same work, formally contradicts the model differential equation. Such a successful agreement of the theory developed in [1] with the experiment, which can be credited to the authors of this work, is explained by the selection of approximating functional solutions that have no basis.

Approximate solution of equation (5) By the method of successive approximations With additional assumptions

The previously obtained analytical solution (34) of equation (5) leads to quadratures of special functions, so there is interest in an approximate solution of this equation by the method of successive approximations by differentiation. This method has some advantages in solving this equation, which describes a physical phenomenon, but in some cases the method can lead to divergent solutions.

Let us introduce the following notation:

$$f(\eta)=\frac{\kappa^3}{\sqrt{\psi\zeta}}. \quad (39)$$

As a first approximation, we can put (κ' is a constant), that:

$$\frac{d\varphi}{d\eta}=\frac{1}{\kappa'}\frac{1}{\eta}. \quad (40)$$

Differentiating (39) with respect toη, we get:

$$\frac{d^2\varphi}{d\eta^2}=-\frac{1}{\kappa'}\frac{1}{\eta^2}. \quad (41)$$

If we substitute (40) and (41) in equation (5), then we can see that the term determining the shear stresses in this case has become negative. The latter contradicts the physical meaning, so for this method it should be changed to positive. In this case, there is a shortcoming directly in the formulation of the problem being solved.

With a direct numerical solution of equation (5), the above-discovered shortcoming of the model cannot be revealed.

If we substitute in (5) positive expressions for derivatives (40) and (41), then we can obtain the following:

$$\frac{d\varphi}{d\eta} = \frac{1}{1+f(\eta)\frac{\eta}{\kappa'}}. \qquad (42)$$

Now you should getthe second positive derivative by differentiating expression (42):

$$\frac{d^2\varphi}{d\eta^2}$$

$$\frac{d^2\varphi}{d\eta^2} = \frac{\eta\frac{df(\eta)}{d\eta}+f(\eta)}{\kappa'\left[1+f(\eta)\frac{\eta}{\kappa'}\right]^2}. \qquad (43)$$

The next approximation is obtained by substituting (42) and (43) into equation (5):

$$\frac{d\varphi}{d\eta} = \frac{1}{1+\frac{f(\eta)}{\kappa'}\frac{\eta\frac{df(\eta)}{d\eta}+f(\eta)}{\left[1+f(\eta)\frac{\eta}{\kappa'}\right]^2}\eta^3}. \qquad (44)$$

The last expression can be enough, after whichis to find the function f(η) and a constantκ', based on the asymptotic boundary conditions (21) and (22), and by putting forward the hypothesis (a=const):

$$f(\eta) = a(f(\eta))^{1/2}. \qquad (45)$$

Substituting (45) into (44), we obtain:

$$\frac{d\varphi}{d\eta} = \frac{1}{1+\frac{\frac{3}{2}\frac{a^2}{\kappa'}\eta^4}{\left[1+\frac{a}{\kappa'}\eta^{3/2}\right]^2}}. \qquad (46)$$

Determining the constants from the above asymptotic boundary conditions, we obtain:

$$\frac{d\varphi}{d\eta} = \frac{1}{1+\frac{a_1\,\eta^4}{\left(1+a_2\eta\sqrt{\eta}\right)^2}}, \qquad (47)$$

Where a1=2.36172·10–4, a2=2.42979·10–2.

Solution of the last equation in quadratures:

$$\varphi = \int_0^{\eta}\frac{d\eta}{1+\frac{a_1\,\eta^4}{\left(1+a_2\eta\sqrt{\eta}\right)^2}}, \qquad (48)$$

The numerical solution of this integral is shown in Figure 1. It is easy to see that this solution correlates well with classical experimental data, for example, Laufer, Reithard, Bauman Moscow State Technical University [7], etc.

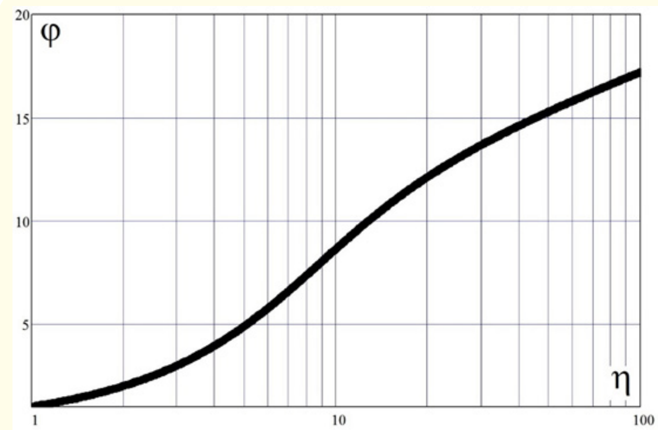

Figure 1: Theoretical profile of dimensionless velocity along the thickness of the boundary layer in turbulent flow based on the numerical solution of integral (48).

For the last integral for small values, an approximate analytical approximation can subsequently be obtained by adopting:

$$\varphi = \int_0^{\eta}\frac{d\eta}{1+\frac{a_1\,\eta^4}{\left(1+a_2\eta\sqrt{\eta}\right)^2}} \cong \int_0^{\eta}\frac{d\eta}{a_3+a_4\,\eta^2} \qquad (49)$$

The integral (49) can be expressed analytically:

$$\varphi \cong \int_0^{\eta}\frac{d\eta}{a_3+a_4\,\eta^2} = a_5\cdot \mathrm{arctg}(a_6\,\eta), \qquad (50)$$

Where; $.a_5 = \frac{1}{\sqrt{a_4 a_3}} = 11{,}15 a_6 = \sqrt{\frac{a_4}{a_3}} = 0{,}097$

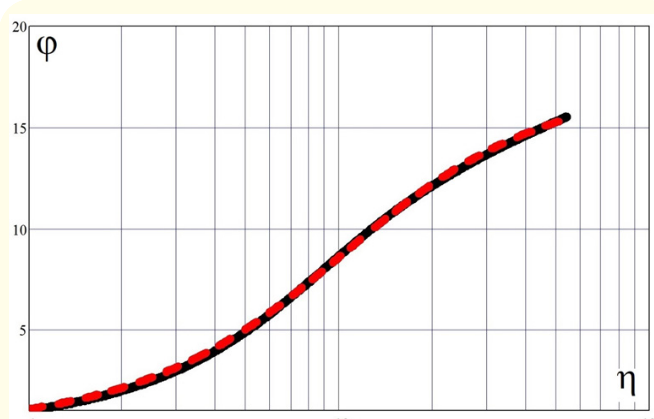

Figure 2: Comparison of the calculated data using the approximate formula (50) (dashed line) with the solution obtained by numerical integration (48) (solid line) relative to the theoretical profile of the dimensionless velocity over the thickness of the boundary layer in turbulent flow.

A comparison of the calculated data using the approximate formula (50) with the solution obtained by numerical integration (48), shown in Figure 2, shows their almost complete identity in the approximated range ().φ ≤ 50.

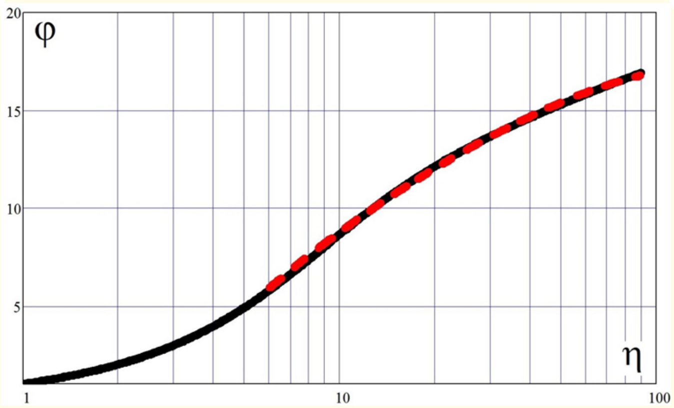

Figure 3: Comparison of the approximate solution (48) (solid line) with the direct numerical solution of equation (5) (dashed line) when specifying the boundary conditions at the point φ = 6.

Direct numerical solution of equation (5) is rather unstable. As an illustration, Figure 3 shows a comparison of approximate solution (48) with direct numerical solution of equation (5) when setting boundary conditions at the point η = 6. As can be seen from Figure 3, the correlation of the above solutions is very good, which indicates the adequacy of the proposed solution by the method of successive approximations.

Therefore, the approximate solution obtained in this work, obtained using the method of successive approximations and using functional asymptotic boundary conditions, can be applied in practice, since they are in good agreement with both the experiment and the direct numerical solution of the basic differential equation.

Key Findings

- In this paper, an exact analytical solution is found for the differential equation for tangential stresses in a turbulent boundary layer, which is a special case of the so-called Abel differential equation of the second kind.
- The solution of the differential equation for shear stresses in a turbulent boundary layer was obtained using the special Lambert function, while it was previously believed that it was not solvable in quadratures. (The proof of the necessity of using the special Lambert function to obtain quadratures is given in the article.)
- In addition to solving the differential equation for shear stresses in a turbulent boundary layer using the special Lambert function, several other important solved special cases of this equation were obtained.
- The advantage of the exact analytical solution of the differential equation for shear stresses in a turbulent boundary layer found in the work is that previously there were either numerical or approximate (for example, by the method of successive approximations by differentiation [1]) solutions to the problem.
- The obtained solution of the differential equation for tangential stresses in a turbulent boundary layer made it possible to identify certain contradictions in the functional boundary conditions used in [1].
- The obtained solution of the differential equation for shear stresses in a turbulent boundary layer in dimensionless form represents a theoretical profile of dimensionless velocity across the thickness of the boundary layer during turbulent flow in the boundary layer.
- In the article, an approximate analytical integral-approximation solution was also obtained, obtained using functional asymptotic boundary conditions by means of the method of successive approximations, which agrees well with both the direct numerical solution of the basic differential equation and with classical experimental data.

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