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Review Article

Twelve Empirical Formulas for Reducing the Closed Motions of a Ray of Light in a Mirror Ellipse to the Case of a Circle

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Abstract

All closed normal movements of a ray of light in the mirror ellipse were classified by reducing them to regular polygons in the circle. This reduction leads to a collection of 12 related formulas based on just two integers: 2 and 14. To transition from a normal E-polygon (EP) to an arbitrary one, a new parameter t was used (which means, in essence, a normalized time).

Keywords: E-Polygon; Mirror Ellipse

Background

Everyone knows what a regular polygon in planimetry is. See Figure 1.

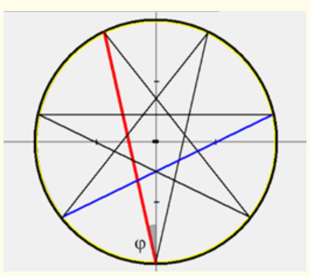

Figure 1: A regular polygon n = 7 and r = 3. (7 - number of sides; 3 - number of revolutions around the center of the circle).

Let's take another equivalent definition for it. Let's imagine that our circle is mirrored from the inside and launch a ray of light into it. If through n reflections it goes to the initial, then the figure described by it is a regular polygon.

Let's take this definition as a basis and set the following problem: what a "regular polygons" (further – EP) exists in the ellipse and what is their classification.

The seeking of the normal EPs

We will look for the EPs in the ellipses in two stages. First of all, we will find all the normal EP, after which we will find all the others.

Definition: normal EPs are those that contain point B among their vertices (see Figure 2).

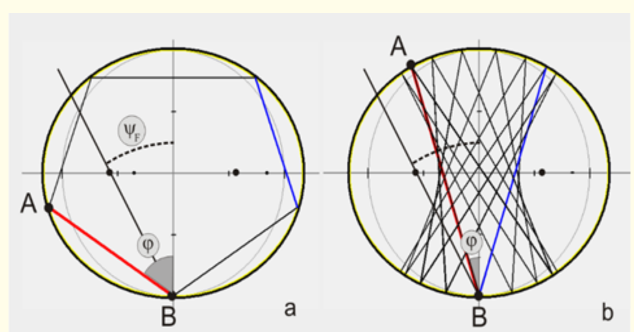

Figure 2: 2a is the 1st order EP (j<yF); 2b is the 2nd order EP (j>yF).

As stated in [1], all sequences of rays in an arbitrary ellipse are divided into two groups: those that do not intersect the segment between the foci (all), and those that intersect (also all). Obviously, all EP are divided into the same two large groups.

We scanned the circle and various ellipses along the angle j (10^o < j < 80^o) to find the minimum of the expression $(A_x-A_x^{(n)})^2 + (A_y-A_y^{(n)})^2 + (B_x-B_x^{(n)})^2 + (B_y-B_y^{(n)})^2$. (A and B are the coordinates of the initial beam; $A^{(n)}$ and $B^{(n)}$ are the coordinates of the n times reflected beam). The found minima (interpreted as the desired EP) turned out to be very pronounced and numerous. Our task was to describe them all correctly and classify.

Normal EP in a circle (regular polygons)

Figure 3 shows a plot of 1/n (n is the number of sides of the EP, in this case the number of sides of a regular polygon) from j (Figure 1-2) for all EPs with n < 100. (Hereinafter, until the contrary is stated, EP means normal EP, that is, it passing through point “B” in Figure 1-2).

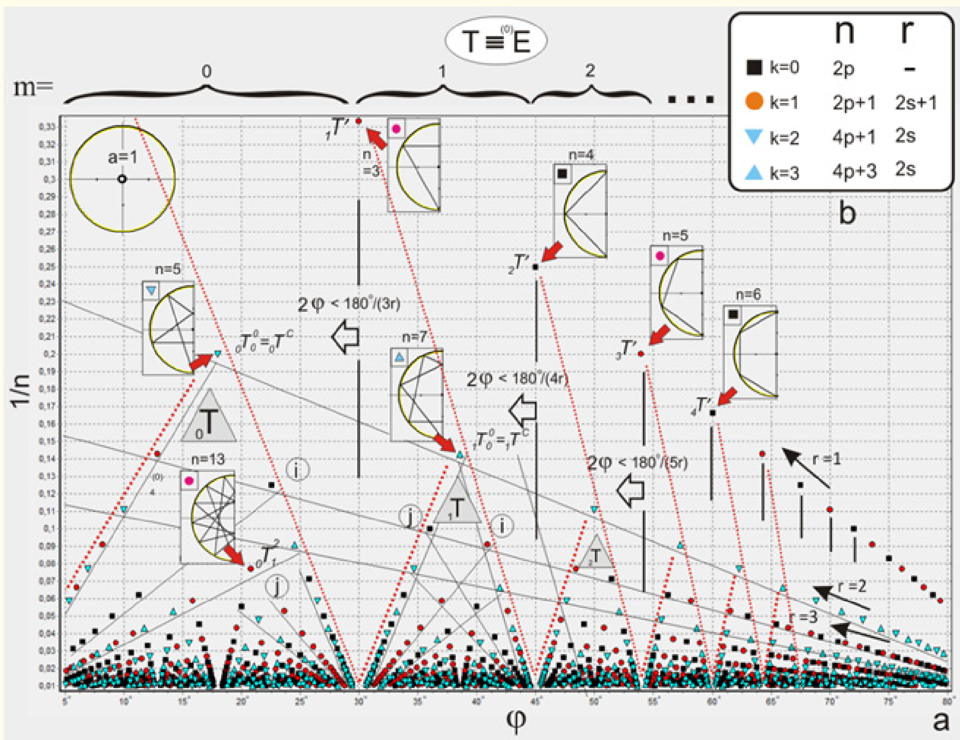

Figure 3: 3a is a graph of 1/n (n is the number of EP sides) versus normal angle j in degrees. The graph is a series of decreasing in size cluster triangles (CTs) (0, 1, 2, ...) lying in the corresponding zones according to j (j<60o; 45o<j<60o; 45o <j<36o; ...) separated by EP with r=1. (These EP are called leading for a given cluster triangle and are indicated by a sign “”). This graph is good because all EPs (regular polygons) are located on it along their lines. Each point on the graph (specific EP) has its own unique numbers n (number of sides) and r (number of revolutions of EP around the center). Black squares represent EP with n=2p (k=0; k- kind of EP Table 1). Red circles indicate EP for odd n and r (n=2p +1; r=2u+1) (k=1). Turquoise triangles with vertices upward are indicated by EP with r=2u and n=4p+1 (k=2). Turquoise triangles with vertices down are indicated by EP with r=2u and n=4p+3 (k=3). Figure 3b is a schematic representation of this.

Let’s take a closer look at what is the cluster triangle of EP (CT) in the graph itself.

Cluster triangle (CT) and its leading EP

The CT cluster triangle consisting of EPs always exists in some series; serial number of the series is m (in Figure 3 m = 0, 1, 2…).

We enter the coordinates i and j as shown in Figure 4a and write out (from Figure 3) the three CT with m = 1, 2, 3 (ignoring CT with m = 0) with its first values for functions k and n of EP. See Figure 4a.

CT is located on a certain “canvas” which is a well-known Pascal [2] triangle, rarefied by numerous “holes”, in places in which EPs are absent.

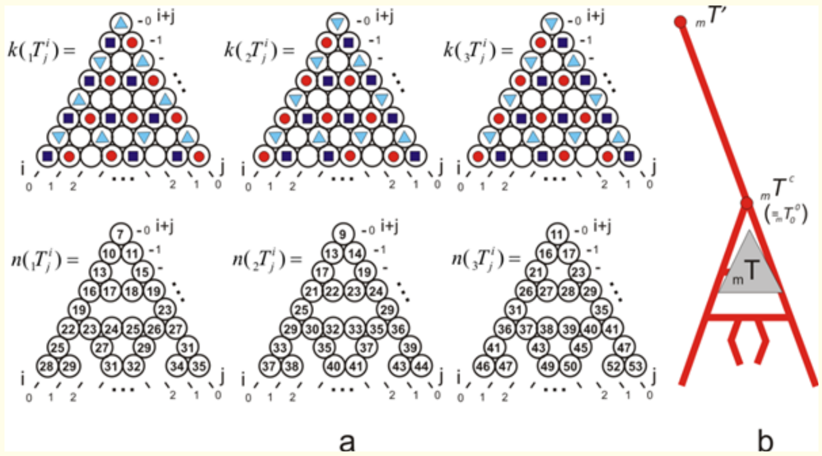

Figure 4: 4a – CT from Figure 3 for m = 1 and m = 2 m = 3 without its leading EP. 4b – schematic designation of CT in figures.

All holes in all CTs are located equally.

Normal EP of arbitrary order in an arbitrary ellipse

Here are the graphs similar to Figure 3 for 11 different ellipses that go with an increase eccentricity. Now, for the first 9 ellipses (Figure 5).

It can be seen that Figure 5 and 6 consist (approximately!) of the same cluster triangles (CT) as the ideal Figure 3, but with the different values (1/n) and (j).

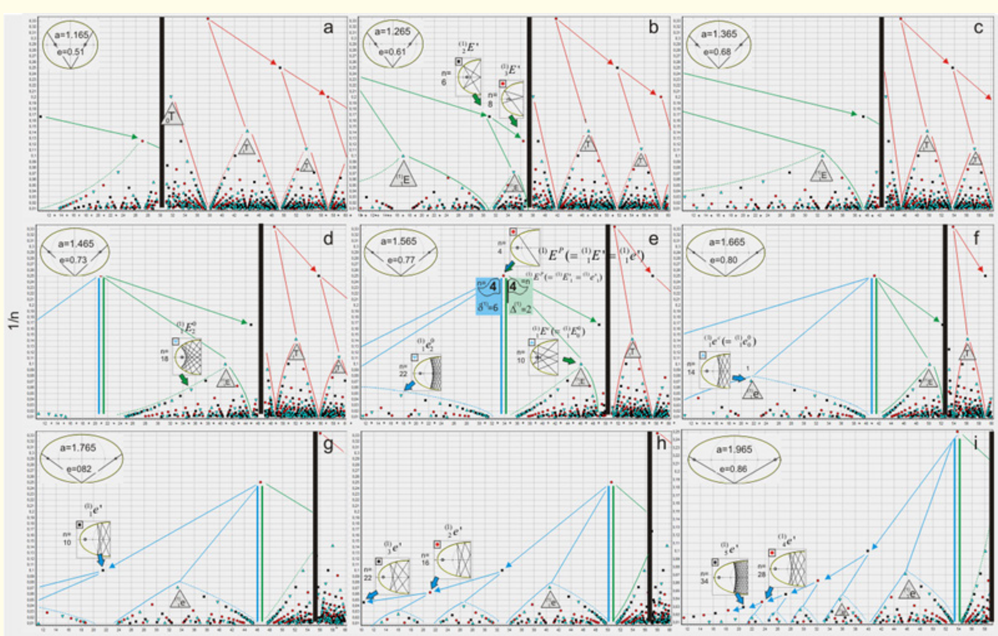

Figure 5: The thick black line indicates the angle yF from Figure 2. To the left of it on each graph it is the movement of rays of the second order, to the right is the first. The definition of k for the 1st order is exactly the same as in Figure 3. For motion of the 2nd order the definition for the function k is the different (explanations is in the following figure). Double colored vertical lines in the Figure 5 indicate our Super-Tops. This is Super-Top number 1…. and then for 2 more (Figure 6b, 6c). The site [3,4] shows the same drawings, but in better resolution.

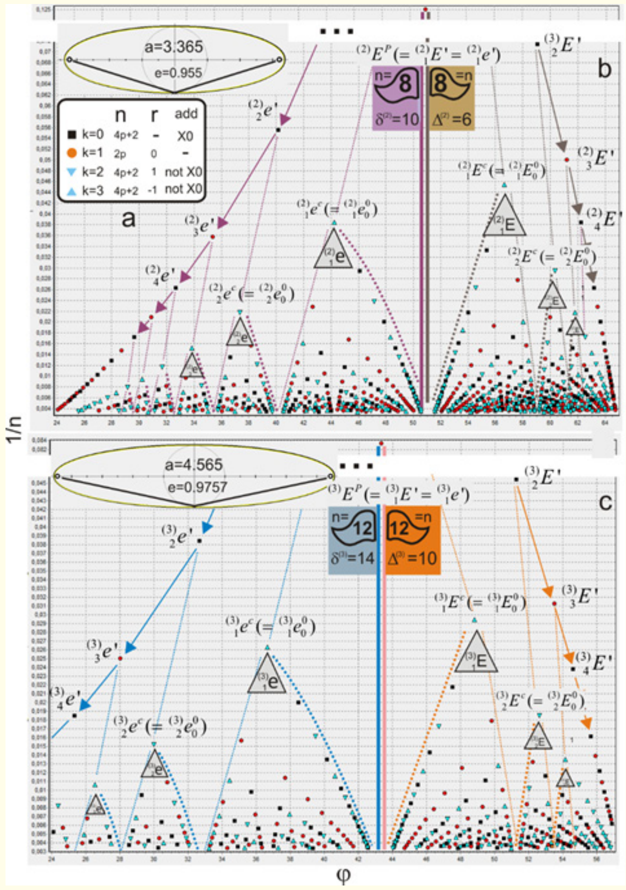

Figure 6: 6a is the definition of the function k of the second order of motion. k=0 – (black square) EP with n=4k+2 and the trajectory of the normal EP rays passes through the center of the ellipse; k=1 – (red circle) n=4k; k=2 – (turquoise triangle vertex down), n=4k+2 and the rays passes does NOT pass through the center and r=1; k=3 – (turquoise triangle vertex up) all the same as for the case k=2 but r= -1. 6b and 6c - continuation of the Figure 5a-5i with increased eccentricities. The dimensions of the ellipses are chosen so that the peaks of the next Super-Tops fall into the approximate center of the drawing.

Function k (kind of EP) for 1st and 2nd order of motion

Let's put together all what we said about the function k into one, general, table 1.

Recall that the type of movement in the first and second order is radically changing. In the first order, these are circular movements around the center of the ellipse. In the second order, it is vibrational movements around point B.

The whole Cartoon in one drawing. EP symbols

Let's imagine an ideal Cartoon, frames from which are our graphs (1/n) x (φ) (but only without restrictions on φ and n; φ from 0^o to 90_o degrees; n – from 3 to ∞), and on the time axis we have the eccentricity that will change from e=0 (a=1) to e=1 (a=∞). Let's try to fit everything that we see in the Cartoon, schematically, into one drawing. See Figure 7.

Our Cartoon at each time for e>0 (a>1) will consist of two parts separated by a vertical line (y_F). On the left of this line there will be the 2nd order EPs, on the right the 1st order EPs. It is logical that we also have two schemes: Figure 7a - for movement of the 1st order; Figure 7b - for 2nd order movement.

Let's start with Figure 7b.

Order of the motions	k (kind) of EP	Number of sides (n)	Number of revolutions around the center of the ellipse	Type of trans-formation at an arbitrary EP	Number and se-quence of colored markers	Example (cartoon)
1	0 ■	2u		1, (Type I)	2; Red, Blue	[5], ${}_0T^1{}_0$
	1 ●	2u+1	2p+1	2, (Type II)	4; Red, Blue, Purple, Green	[6] ${}_1T'$,
	2 ▼	4u+1	2p			[22]
	3 ▲	4u+3	2p			[23]
2	0 ■	4u+2	-	3, (Type III)	3; Red, Blue, Red, Green	[7] ${}^{(1)}{}_2E'$, [17,18]
	1 ●	4u	0	4, (Type IV)	3; Green, Red, Blue, Red	[8] ${}^{(1)}{}_3E'$, [11,13,16]
	2 ▼	4u+2	1	5, (Type V)	3; Red, Green, Red, Blue	[14,15,20]
	3 ▲	4u+2	-1			[9] ${}^{(1)}{}_1E^c$, [12,19,21]

Table 1: Our designations for function k for the 1st and 2nd orders of motion. In bold are those markers where their EP is in a shimmering phase.

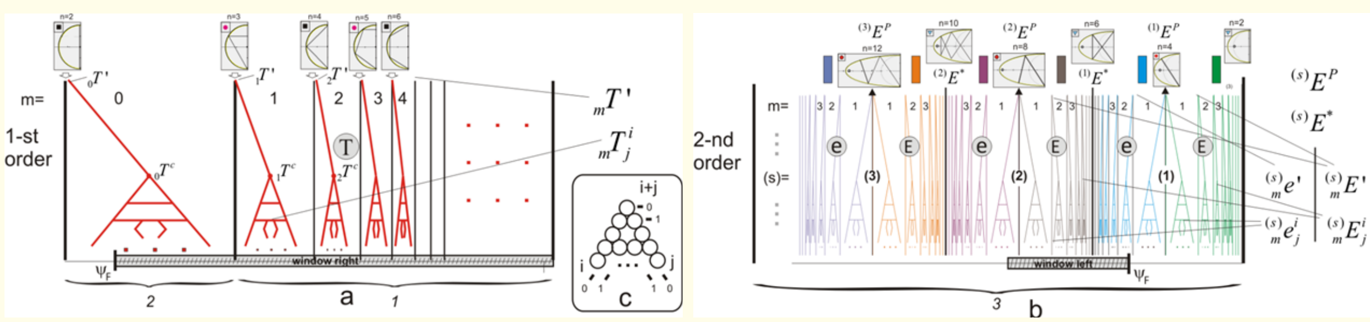

Figure 7: Is our Cartoon represented by one diagram. The figure consists of two parts: 7a is a diagram for the first order of movement; 7b is a diagram for the second order of movement. A shaded rectangle shows what exactly hits the screen for a certain "random" frame. The letters in the figure: (s) - Super-Tops number; m - number of CT in the series; i and j are the coordinates of the particular EP in CT ("Pascal's triangle"). Further explanations in the text...

Our diagram is a sequence of going to infinity of what we call Super-Tops. They are numbered by a letter (s). EP located in the peaks of Super-Tops bear their designation ${}^{(s)}E^P$.

The Super-Tops are separated by free-standing EP designated as ${}^{(s)}E^*$.

Each Super-Top falls down both to the right and to the left by the series of the cluster triangles (CT).

For those EP that are located to the left of the Super-Tops a small letter "e" is used when writing, for those that are located to the right – a large letter "E".

The corresponding CT series is written as T (1^{st} order) and ${}^{(s)}E$ or ${}^{(s)}e$ (2^{nd} order).

The cluster triangles (CTs) themselves are written as ${}_mT$ (1^{st} order) and ${}^{(s)}_mE$ or ${}^{(s)}_me$ (2^{nd} order). ("s" is the Super-Tops number, "m" is the cluster number in the series. s, m=1, 2 ...).

The EP themselves are written as ${}_mT^i_j$ (1^{st} order) and ${}^{(s)}_mE$ or ${}^{(s)}_me$ (s) (2^{nd} order); where i and j are the coordinates in "Pascal's triangle" (Figure 4a, 7).

The leading EP in each CT are indicated by a dash. ${}_mT'$ (1^{st} order) and ${}^{(s)}_mE'$ or ${}^{(s)}_me'$ (2^{nd} order).

EP in each cluster (CT) having coordinates i,j = 0 have the additional name ${}_mT^c$ ($={}_mT^0_0$; 1^{st} order) and ${}^{(s)}_mE^c$ or ${}^{(s)}_me^c$ ($={}^{(s)}_mE^0_0$ or $={}^{(s)}_me^0_0$; 2^{nd} order).

(We will use 2 types of designations of our objects: using lower and upper registers and without them, in a string. In this case, our designations are enclosed in brackets different for different categories: "[]" - for the series of CT, "< >" - for CT themselves, "()" - for EP. Examples: ${}^{(2)}_3E \equiv$ <2,3,E>, ${}^{(3)}E^P \equiv (3,E^P)$, ${}^{(3)}_1e^2_2 \equiv$ (3,1,e,2,2)).

We declare that no other normal EP exists!

We give the following formulas from Figure 5, 6, 7.

$k({}^{(s)}E^*) = 2; \quad n({}^{(s)}E^*) = 4s+2;$ ------- (1)

$k({}^{(s)}E^P) = 1; \quad n({}^{(s)}E^P) = 4s;$ -------- (2)

$k({}_mT') = m\%2; \quad n({}_mT') = 2+m;$ ----------------(3)

$k({}^{(s)}_mE') = m\%2; \quad n({}^{(s)}_mE') = 4s+\Delta^{(s)}m; \quad \Delta^{(s)} = 2+4(s-1);$ - - (4)

$k({}^{(s)}_me') = m\%2; \quad n({}^{(s)}_me') = 4s+\delta^{(s)}m; \quad \delta^{(s)} = 14+4(s-1);$ --- (5)

$k({}^{(s)}_mE^c) = 2+m\%2; \quad n({}^{(s)}_mE^c) = n({}^{(s)}_mE') + n({}^{(s)}_{m+1}E');$ -------- (6)

$k({}^{(s)}_me^c) = 2+m\%2; \quad n({}^{(s)}_me^c) = n({}^{(s)}_me') + n({}^{(s)}_{m+1}e');$ ---------- (7)

$k({}_mT^c) = 2+m\%2; \quad n({}_mT^c) = n({}_mT') + n({}_{m+1}T');$ (8)

It can be seen that all formulas are based, in fact, on two integers 2 (formula 4) and 14 (formula 5).

Formula n(EP) from i and j

Figure 8 shows functions **k** and **n** for i,j < 4, and CT equal to <1,1,E>, <1,1,e>, <1,2,E>, <1,2,e>, <1,3,e>, <2,1,E>, <2,1,e>, <2,2,E>, <2,2,e>, <2,3,E>, <2,3,e>, <3,1,E>, <3,1,e>, <3,2,E>, <3,2,e>, <3,3,E>, <3,3,e>; (${}^1_1E$, ${}^1_1e$, ${}^1_2E$ ...).

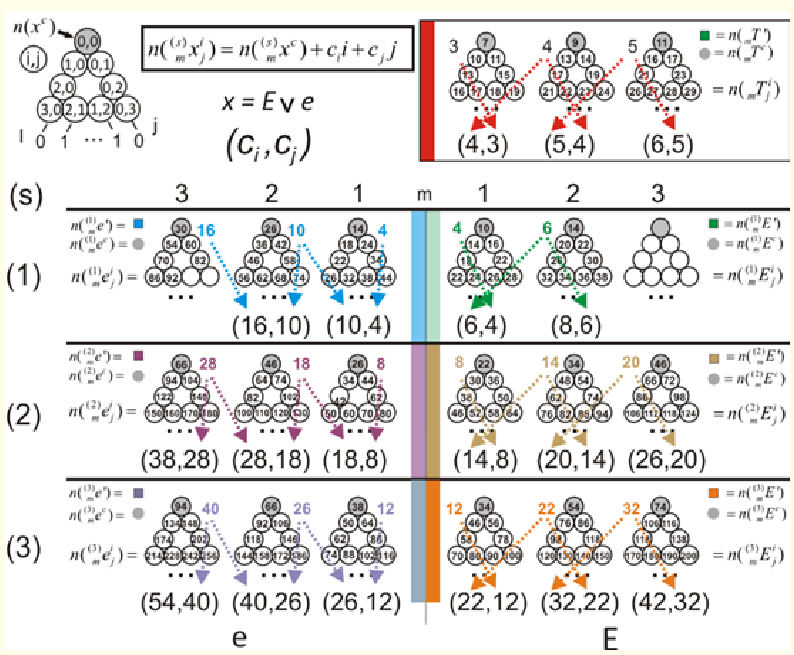

Figure 8: Illustration of formulae (9), (10), (11).

From Figure 8 follows…

$$n({}_mT^i_j) = n({}_mT^c) + n({}_{m+1}T')*i + n({}_mT')*j \text{ ------ (9)}$$

$$n({}^{(s)}_mE^i_j) = n({}^{(s)}_mE^c) + n({}^{(s)}_{m+1}E')*i + n({}^{(s)}_mE')*j \text{ ------- (10)}$$

$$n({}^{(s)}_me^i_j) = n({}^{(s)}_me^c) + n({}^{(s)}_{m+1}e')*i + n({}^{(s)}_me')*j \text{ ------------- (11)}$$

We give the last 12 formula: for any (s) m, i, j true

$$k({}_mT^i_j) = k({}^{(s)}_mE^i_j) = k({}^{(s)}_me^i_j) \text{ -------- (12)}$$

Figure 9 illustrates this for the case s=1, m=1, i+j<8.

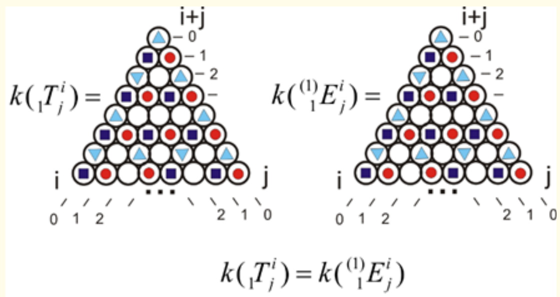

Figure 9: Illustration of the formula 12.

"Birth" and "death" of EP

In terms of EP "birth" (appearance) in the Cartoon and EP "death" in the Cartoon, all EPs are divided into three categories (shown in the diagram (Figure 7) with curly brackets and numbers 1, 2, 3).

At the very beginning (e=0), all first-order EPs are present, and there are no second-order EPs.

Immediately after the start of the Cartoon, in the second order, a small interval will appear on the screen (in the diagram, from the left end), which will immediately begin to move away from the border (y_F) in the direction to the left (we did not observe this, but it is difficult to imagine some other behavior).

In the first order, at the same time, part of the CT ${}_0$T will begin to gradually disappear, which at infinity (at e=1) will disappear completely.

Thus, all CT are divided into three groups on the issue of "birth and death".

1st group ("immortals") – time of birth e=0, time of death e=1. It is all EPs contained in CT= ${}_1$T, ${}_2$T, ${}_3$T…

2nd group ("immortals only if time is run the other way") – time of birth e=0, time of death e=e_D. It is EPs contained in CT=${}_0$T.

3rd group ("mortals") – time of birth e=e_B (which is strictly greater than 0) and time of death e=$_{eD}$ (which is strictly less than 1). All 2nd- order EPs belong to this group.

All EPs from birth to death behave well and correctly, clearly taking their place in their cluster. What do we mean?

Empirically established the following fact! (? Was not strictly checked on the computer!).

Let given EP= ${}^{(s)}_mX^i_j$ (X=E or e or T). $j_e({}^{(s)}_mX^i_j)$ function j for an ellipse with eccentricity e. And let for some j1, j2, and $e_B<e<e_D$ $j_e({}^{(s)}_mX^i_{j1}) > j_e({}^{(s)}_mX^i_{j2})$.

Then it is argued that then this inequality will be true for all $e_B<e<e_D$ (of course, when both EPs are present in the frame)!

The same statement is true, as well, for the inequality regarding i: $j_e({}^{(s)}_mX^{i1}_j) > j_e({}^{(s)}_mX^{i2}_j)$.

Change from Normal EP to General EP

To change from normal EP to general EP we must add a new parameter t to normal EP.

"t" – is a real number from 0 to 1. (Basically, it's just - TIME, the normalized frame number in the cartoon. Now we are talking about another "cartoon". We have already completed everything with the main Cartoon).

Let's take any normal EP and choose a small offset to the beginning of the beam (dA) and to the end of the beam (dB) (replace the original beam AB with A'B'). Then we arrange the markers on the ellipse (see below), and launch our new "cartoon with a small letter".

First frame: normal E=EP. Second frame: EP after n-reflections - R(E). The third frame is EP R(R(E)) and so on... $R^k(E)$. (In essence, you need to make k*n reflections, but draw only the last n rays). See [5,22]. And our cartoon will consistently describe all general EPs tied to this normal! Moreover, it will do this in the same time from one color marker to the next. See Figure 10.

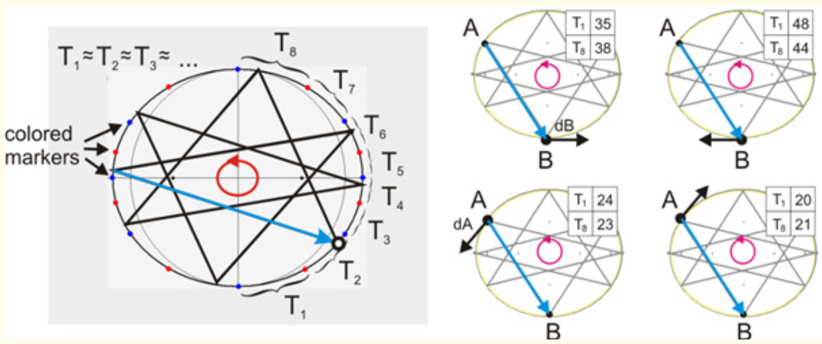

Figure 10: T – is the number of frames from one color marker to another.

Which dA and dB offsets do you need to select for this? Our answer is ANY. The main thing is that they are small. But the problem is that my computer cannot make them very small, the program begins to work wrong. But in fact, this is NOT NECESSARY! Moderate accuracy ~ 1000 reflections per full revolution (or ~ 100 reflections per passage of one marker) is quite suitable.

Fig,10 shows how the difference between T_8 and T_1 ($E={}_0T^1{}_0$, n=8; r=3) changes at completely different initial offsets dA and dB. It can be seen that the difference is not significant. Even better reduces this difference our voluntaristic choice! For 1st order EP we take dA = dB. For 2nd order EP some dA and dB = 0. The value of dA and dB is just selected to provide ~ 500-1000 reflections per full cycle.

One last thing! Really new will be only EP from the beginning to the first marker! And all the other EPs are their different reflections.

Let's talk about what markers and how we need to place them?

General 1st order EP (Types I, II)

The four kinds of normal 1st order EPs (k=0..3) are divided into two types of possible transformations: Type I (k=0) and Type II (k=1..3). Figure 11a and Figure 11b.

The current position of point B is called the control marker. Let t be the "time" (frame number); u=[t/T], t= {t/T} (where T is the

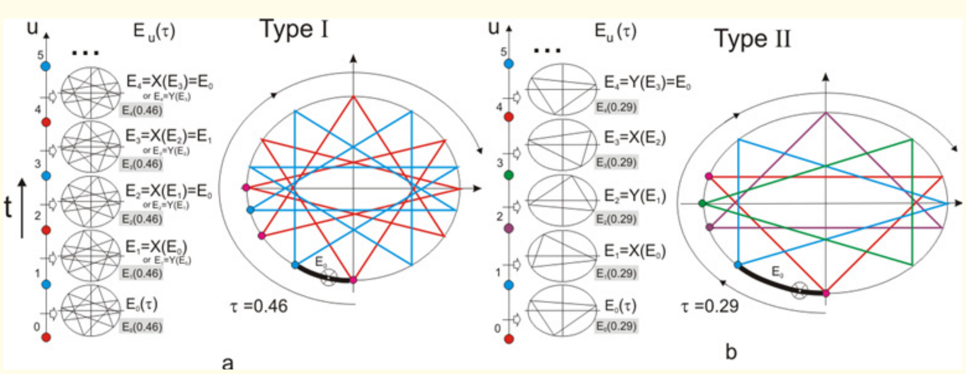

Figure 11: 11a – EP E=0T10 (a=1.165; j=33.38 o; n=8; r=3; k=0); 11b – EP E=1T' (a=1.365; j=45.44o; n=3; r=1; k=0). X (E) and Y (E) are the symmetrical EPs about the X and Y axes.

time of the control marker passage from one color marker to the next. Then the EP at time t is denoted by $E_u(t)$.

In the next two chapters we will tell you how to place markers and what colors they have.

Type I

See [5]. (A control marker (the current position of point B) is marked in the cartoon with a moving black circle. Unlike stationary, colored ones).

The vertices of a normal EP are marked in red. This EP is symmetrical about X and Y. After time T this EP passes into a new EP with the same symmetries. Its vertices are marked in blue. Thus, the sequence of markers through which the control marker will pass: red-blue-red-blue... and so on.

Type I EP will always maintain central symmetry. On Figure 11a the control marker has made a complete revolution in 16 cycles of T. 16=n*2.

Type II

See [6].

The vertices of a normal EP are marked in red. The vertices of its X-symmetrical version are marked in purple. Through after T reflections, the EP becomes X-symmetric. Color of these vertices is blue. Y-symmetric vertices to it will be painted green. Then the sequence of markers passed by control marker going around the center will be red-blue-purple-green ... and again red-blue-purple-green ... and so on.

On Figure 12b the control marker has made a complete revolution in 12 cycles of T. 12=n*2.

The time T when moving from one marker to another will be the same (as T tends to infinity). In addition, as shown in the figure, all subsequent EPs $E_u(t)$ u>0, are expressed exclusively from the EP $E_0(t)$ by reflections relative to the X axis or relative to the Y axis. That is, this interval alone is enough to describe all general EP.

General 2nd order EP (Types III, IV, V)

In the second order, we will have three changes relative to the first.

First, the control marker will no longer make a complete revolution around the center of the ellipse, but will oscillate around the starting point B.

Secondly, in its transformation, the EP will pass through the so-called "flickering" phases, when the vertices of the EP partially coincide and the number of sides is actually halved. In Table 1, in the column "Number and sequence of markers", the color of the marker, when this happens, is given in bold.

Thirdly, a local time reversal operation (E(t)=>E(1-t)) will be added to the reflections of the EP relative to the axis X and Y (X (E), Y (E)). Let's number the passage of color markers with a control marker. Then for any even pass this will be the local time reversal of the previous pass plus, may be, reflection along the X or Y axis. See Figure 12.

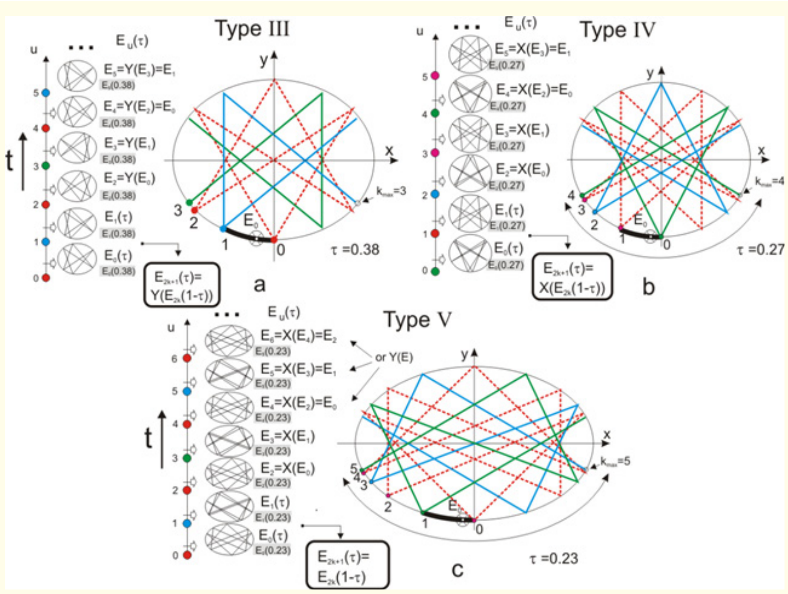

Figure 12: 12a – EP E=(1)2E' (a=1.265; j=30.38 o; n=6; k=0; Figure 5b); 12b – EP E=(1)3E' (a=1.165; j=27.18o; n=8; k=1; Figure 5a); 12c – EP E=(1)1Ec (a=1.565; j=46.12o; n=10; r=-1).

The four kinds of normal 2nd order EP (k=0..3) are divided into three types of possible transformations: Type III (k=0), Type IV (k=1) and Type V (k=2,3). See Figure 12.

Type III

See [7].

The normal EP is in an ordinary (not "flickering") phase. It has X and Y-symmetry. Mark its vertices with a red marker. At time T, the EP will go into a "flickering" X-symmetric phase. Its vertices will be marked with a blue marker. The vertices of the EP Y-symmetrical to it will be marked green. Then the sequence of markers through which the control marker passes is red-blue-red-green- red-blue-red-green... and so on. After counting n/2 (n/2=3; Figure 13a) markers, the control marker will reach the maximum on the Y-axis and move back

Type IV

See [8].

The normal EP is in a "flickering" phase. It has the Y-symmetry. Mark its vertices with a green marker. Vertices of X-symmetric EP to it marked with a blue marker. At time T, the EP will have X and Y symmetry. Its vertices will be marked with a red marker. Then the sequence of markers through which the control marker passes is green-red-blue-red- green-red-blue-red... and so on. After counting n/2 (n/2=4; Figure 13b) markers, the control marker will reach the maximum on the Y-axis and move back. This will not affect the sequence of markers in any way.

Type V

See [9].

The normal EP is in an ordinary phase. It has X and Y-symmetry. Mark its vertices with a red marker. At time T, the EP will go into a "flickering" central-symmetric phase. Its vertices will be marked with a green marker. The vertices of the EP X-symmetrical (or Y-symmetrical, what is the same) to it will be marked blue. Then the sequence of markers through which the control marker passes is red-green-red-blue- red-green-red-blue... and so on. After counting n/2 (n/2=5; Figure 12c) markers, the control marker will reach the maximum on the Y-axis and move back.

The expression of $E_u(t)$ through $E_0(t)$ (u>0) is shown in the Figure 12.

Let us summarize: in all five cases of types (Type I, Type II, Type III, Type IV, Type V) all EPs are the corresponding reflections with respect to X, Y and time of $E_0(t)$.

Conclusion

Although we do not strictly prove formulas (1)-(12), their correctness is beyond doubt. Anyone can download the program [22] and, according to the instructions, easily draw up a graph similar to Figure 5-6 for any ellipse and check the correctness of formulas (1)-(12) or write down a standard "confirmation" cartoon similar to [5-24] (markers are placed automatically) and make sure that all the transition times from one color marker to the next are equal to each other.

Bibliography

1. A Kornyushkin. "Three New(?) Properties of an Ellipse and an Ellipsoid of Revolution (Computer Analysis)". *Current Trends in Computer Sciences and Applications* (2022).
2. https://en.wikipedia.org/wiki/Pascal%27s_triangle
3. (9 ellipses).
4. (2 ellipses).
5. 0T10=(0,T,1,0) k=0.
6. 1T'=(1,T') k=1.
7. (1)2E'=(1,2,E') k=0.
8. (1)3E'=(1,3,E') k=1.
9. (1)1Ec=(1,1,E,0,0) k=3.
10. (Type I-V).
11. (1)3E'=(1,3,E') k=1.
12. (1)1Ec =(1,1,E,0,0) k=3.
13. (1)EP=(1,E') k=1.
14. (1)E*=(1,E*) k=2.
15. (2)E*=(2,E*) k=2.
16. (2)EP=(2,E') k=1.

17. (1)2E'=(1,2,E') k=0.
18. (1) 2e'=(1,2,e') k=0.
19. (2) 1ec=(2,1,e,0,0) k=3.
20. 2Tc=(2,T,0,0) k=1.
21. 1Tc=(1,T,0,0) k=3.
22. (1)1E34=(1,1,E,3,4) k=3.
23. (Program E-poly.exe).
24. Find regular sequences in circle.
25. Find regular sequences in ellipse (a=1.165).