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Review Article

Exact solution of Klein-Gordon Equations Using Homotopy Perturbation Method and the Variational Iteration Method

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Abstract

This article investigates the Klein-Gordon equations (KGEs) and utilizes both the coupling the variational iteration method (VIM) and homotopy perturbation method (HPM) to derive precise solutions. Theoretical insights are integrated with these analytical approaches, providing a deeper comprehension of the underlying mathematical principles. Numerous illustrative examples are showcased to underscore the methods' efficacy and dependability. The obtained outcome highlights the simplicity and efficiency of the proposed techniques in solving KGEs. Through this analysis, the paper contributes valuable insights into the application of coupling HPM and VIM in addressing the complexities of KGEs, underscoring their potential for tackling challenges in diverse scientific domains.

Keywords: Klein – Gordon; VIM; HPM

Introduction

The KGEs represent a fundamental set of PDEs that play a crucial role in various branches of theoretical physics, including quantum field theory, condensed matter physics, and relativistic quantum mechanics. These equations describe the behavior of scalar fields, and obtaining exact solutions is paramount for understanding the underlying physical phenomena associated with these fields. This article initiates a thorough investigation into the precise solutions of Klein-Gordon equations (KGEs), utilizing two potent analytical approaches: the variational iteration method (VIM) and the homotopy perturbation method (HPM). The variational iteration method is a flexible analytical technique that has garnered recognition for its effectiveness in addressing diverse sets of differential equations. It is rooted in the concept of constructing a correctional functional, incorporating an auxiliary parameter, and iteratively refining the solution. The VIM has proven successful in tackling nonlinear problems, making it an attractive choice for exploring the intricate nonlinearities embedded in the KGEs. The HPM introduces a homotopy parameter to smoothly deform a known solution towards the desired solution. This method harnesses the power of perturbation theory to iteratively improve the approximation, allowing for a systematic approach to solving nonlinear differential equations. The HPM has demonstrated efficacy in addressing problems with nonlinearities and has been applied to various scientific and engineering domains. The combination of the VIM and the HPM provides a robust approach to tackle the challenges posed by the Klein-Gordon equations. The synergy of these methods not only enhances the accuracy of the solutions but also offers a deeper understanding of the underlying physical principles governing the scalar fields described by the KGEs. Theoretical considerations play a crucial role in this exploration, as they provide the necessary foundation for the analytical methods employed. Discussions on the mathematical principles involved in the VIM and HPM contribute to the overall rigor of the analysis. These theoretical considerations are essential for researchers and practitioners seeking to grasp the intricacies of the methods and their application to KGEs. To illustrate the capability and reliability of the proposed methods, we present a series of examples that showcase the obtained exact solutions. These examples serve as benchmarks, allowing for a qualitative and quantitative assessment of the effectiveness of the VIM and the HPM in solving KGEs. The results not only affirm the accuracy of the solutions but also highlight the simplicity and practical applicability of the proposed methods. In summary, this paper embarks on a comprehensive exploration of the exact solutions of KGEs by employing the VIM and HPM. The mathematical complexity of the KGEs necessitates powerful analytical tools, and the synergy of these two methods offers a systematic and effective approach. Theoretical considerations provide a solid foundation for the analysis, and illustrative examples demonstrate the capability and reliability of the proposed methods. Through this exploration, we aim to contribute valuable insights into the analytical solution of Klein-Gordon equations, advancing the understanding of scalar fields in theoretical physics. Multiple effective mathematical techniques, including the VIM [1], the HPM [2], the new iterative method [3], and the ADM [4], have been demonstrated to be effective in resolving partial differential equations as well as algebraic, differential, integro-differential, and differential delay problems. The homotopy perturbation technique

(HPM), introduced by [5,6], has become widely utilized for solving integral equations prevalent in real-world modeling, such as those related to thin film flow and heat transfer [7-19]. HPM operates on the premise that the solution can be expressed as an infinite series, rapidly converging to the exact solution. This study conducts a comparative analysis between the Variational Iteration Method (VIM) and HPM in the context of Klein-Gordon equations [20], providing illustrative examples. Both methods, recognized as powerful and efficient techniques for solving linear inhomogeneous differential equations [21-27], yield rapidly converging, realistic series solutions in practical physical scenarios. VIM simplifies differential equations into manageable sets of ordinary differential equations, reducing computational complexity. The research demonstrates that these techniques demand less computational effort than existing methods, offering quantitatively reliable results. The substantial agreement between numerical results obtained through VIM and HPM across all examples underscores the methods' efficiency, extending their applicability

The varational iteration method's basic concept

The VIM is a powerful analytical technique that seeks to iteratively refine an initial approximation by constructing correctional functional, incorporating an auxiliary parameter. The fundamental idea behind VIM lies in formulating an initial guess and iteratively improving it to obtain increasingly accurate solutions to differential equations. By introducing an auxiliary parameter, the method systematically refines the solutions, converging towards an optimal result. VIM has proven effective in in tackling a wide range of differential equations, providing a versatile and systematic approach to solving complex mathematical problems in diverse scientific disciplines.

To clarify the basic ideas of VIM, examine the differential equation that follows,

$\mathbb{N}\mathbb{U} + \mathbb{L}\mathbb{U} = \mathbb{g}(\mathbb{t})$, (1)

Where g(t) is the source inhomogeneous termand L and N are linear and the nonlinear operators, respectively. We can write down correction functional as follows using VIM.

$\mathbb{U}_{\mathbb{m}+1}(\mathbb{X}) = \mathbb{U}_{\mathbb{m}}(\mathbb{X}) + \int_0^{\mathbb{x}} \lambda(\xi)(\mathbb{N}\mathbb{U}_{\mathbb{m}}(\xi) + \mathbb{L}\mathbb{U}_{\mathbb{m}}(\xi) - \mathbb{g}(\xi))d\xi$, (2)

Where λ is a general Lagrangian multiplier that the variational theory allows for an optimal identification of. The m^{th} approximation is indicated by the subscript n and U_n is regarded as a restricted variation i.e $SU_n=0$.

The Homotopy - perturbation method's basic concept

The HPM is a powerful analytical approach that introduces a homotopy parameter into an a preliminary estimate, allowing regarding the systematic perturbation of solutions in nonlinear differential equations. The core concept involves constructing a homotopy equation, representing a ongoing deformation from a known solution to the desired one. Perturbation theory is then applied to iteratively refine this approximation, providing a reliable means to solve nonlinear problems. The strength of HPM lies in its ability to handle complex, nonlinear phenomena by smoothlyin mathematical modeling and scientific research.

To explain this method, let us consider the following function,

$\mathbb{A}(\mathbb{U}) - \mathbb{f}(\mathbb{r}) = 0, \mathbb{r} \in \Omega$, (3)

Utilizing the boundary conditions as,

$\mathbb{B}\left(\mathbb{U}, \frac{\partial \mathbb{U}}{\partial \mathbb{m}}\right) = 0, \mathbb{r} \in \Gamma$, (4)

Where B, A, F(r) and Γ are general differential operator, a boundary operator, a known analytical function and the boundary of the domain Ω, respectively. Generally, the operator A can be divided in to a linear part L and a nonlinear part N(U).

Equation (3) can be written as,

$\mathbb{N}\mathbb{U} + \mathbb{L}\mathbb{U} - \mathbb{F}(\mathbb{r}) = 0$, (5)

By using the homotopy technique, we construct a homotopy $\mathbb{V}(\mathbb{r}, \mathbb{P}): \Omega \times [0,1] \rightarrow \mathbb{R}$ which satisfies,

$\mathbb{H}(\mathbb{V}, \mathbb{P}) = (1 - \mathbb{P})[\mathbb{L}(\mathbb{V}) - \mathbb{L}(\mathbb{U}_0)] + \mathbb{P}[\mathbb{A}(\mathbb{V}) - \mathbb{F}(\mathbb{r})] = 0, \mathbb{P} \in [0,1], \mathbb{r} \in \Omega$, (6)

Or

$\mathbb{H}(\mathbb{V}, \mathbb{P}) = \mathbb{L}(\mathbb{V}) - \mathbb{L}(\mathbb{U}_0) + \mathbb{P}\mathbb{L}(\mathbb{U}_0) + \mathbb{P}[\mathbb{N}(\mathbb{V}) - \mathbb{F}(\mathbb{r})] = 0$, (7)

Where is an embedding parameter, while is an initial approximation which satisfies the boundary conditions.

$\mathbb{H}(\mathbb{V}, 0) = \mathbb{L}(\mathbb{V}) - \mathbb{L}(\mathbb{U}_0) = 0$, (8)

$\mathbb{H}(\mathbb{V}, 1) = \mathbb{A}(\mathbb{V}) - \mathbb{F}(\mathbb{r}) = 0$, (9)

The changing process of p from zero to unity is just of $\mathbb{V}(\mathbb{r}, \mathbb{P})$ from $\mathbb{U}_0$ to $\mathbb{U}(\mathbb{r})$. This is referred to as deformation in topology, while $\mathbb{L}(\mathbb{V}) - \mathbb{L}(\mathbb{U}_0)$ and $\mathbb{A}(\mathbb{V}) - \mathbb{F}(\mathbb{r})$ are considered homotopy according to the HPM. We can first use the embedding parameter $\mathcal{P}$ as a 'small parameter', and assume the solution of (6) and (7) can be expressed as a power series in $\mathbb{P}$,

$\mathbb{V} = \mathbb{V}_0 + \mathbb{P}\mathbb{V}_1 + \mathbb{P}^2\mathbb{V}_2 + \cdots$ (10)

Setting $\mathbb{P} = 1$ yields in the roughly calculated solution of (3),

$\mathbb{U} = \lim_{\mathbb{P} \to 1} \mathbb{V} = \mathbb{V}_0 + \mathbb{V}_1 + \mathbb{V}_2 + \cdots$ (11)

It is evident from this that the homotopy method plus perturbation method, or (HPM), removes the shortcomings of conventional perturbation methods while maintaining all of its benefits. The series (11) is convergent for most cases. However, the convergent rate depends on the nonlinear operator A(V) The second derivative of A(V) with respect to V must be small because the parameter may be relatively large, i.e., P→1.

The norm of $\mathbb{L}^{-1}\left(\frac{\partial \mathbb{N}}{\partial \mathbb{V}}\right)$, must be smaller than one so that series converges.

Utilizations

In this section, we solve the KGEs using VIM.

Example 3.1

Think about the following Klein-Gordon equation

$\mathbb{U}_{tt} - \mathbb{U}_{xx} - 2\mathbb{U} = -2\mathbb{S}\text{inx}\mathbb{S}\text{int}$, (12)

I.C $\mathbb{U}(x,0) = 0, \frac{\partial \mathbb{U}}{\partial t}(x,0) = \mathbb{S}\text{inx}$. (13)

Case I. VIM

The VIM is a potent mathematical technique employed for solving differential equations by iteratively refining an initial approximation through the construction of a correctional functional.

Integral to VIM is the incorporation of an auxiliary parameter, facilitating the systematic improvement of solutions. This method's versatility is evident in its applicability to a broad spectrum of differential equations, offering a systematic and effective means to tackle both linear and nonlinear problems. By providing a structured approach to refining approximations, the VIM stands as a valuable tool in the arsenal of analytical methods, contributing to the solution of intricate mathematical and physical problems. The corrections functional is given by

$$\mathbb{U}_{n+1}(x,t) = \mathbb{U}_n(x,t) + \int_0^t \lambda(\xi)\left(\frac{\partial^2 \mathbb{U}_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \mathbb{U}_n(x,\xi)}{\partial x^2} - 2\mathbb{U}_n(X,\xi) + 2\mathbb{S}\text{inx}\mathbb{S}\text{in}\xi\right) d\xi, \ (14)$$

Finding the lagrange multiplier after making the correction functionally stationary can be done as follows:

$\lambda = (\xi - t)$,

$$\mathbb{U}_{n+1}(x,t) = \mathbb{U}_n(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 \mathbb{U}_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \mathbb{U}_n(x,\xi)}{\partial x^2} - 2\mathbb{U}_n(X,\xi) + 2\mathbb{S}\text{inx}\mathbb{S}\text{in}\xi\right) d\xi, \ (15)$$

Using the selection $\mathbb{U}_0(x,t) = t\mathbb{S}\text{inx}$, gives the successive approximations as,

$$\mathbb{U}_1(x,t) = \mathbb{U}_0(x,t) + \int_0^t (\xi - t)\left(\frac{\partial^2 \mathbb{U}_0(x,\xi)}{\partial \xi^2} - \frac{\partial^2 \mathbb{U}_0(x,\xi)}{\partial x^2} - 2\mathbb{U}_0(X,\xi) + 2\mathbb{S}\text{inx}\mathbb{S}\text{in}\xi\right) d\xi,$$

$$\mathbb{U}_1(x,t) = t\mathbb{S}\text{inx} + \int_0^t (\xi - t)(-\xi\mathbb{S}\text{inx} + 2t\mathbb{S}\text{inx}\mathbb{S}\text{in}\xi)\, d\xi.$$

$\mathbb{U}_1(x,t) = t\mathbb{S}\text{inx} - \frac{t^3}{6}\mathbb{S}\text{inx} + 2\mathbb{S}\text{inx}\mathbb{S}\text{int}$. (16)

You may write the approximate series solution as follows.

$\mathbb{U}(x,t) = \mathbb{S}\text{inx}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right)$, (17)

And the following is the closed form solution,

$\mathbb{U}(x,t) = \mathbb{S}\text{inx}\mathbb{S}\text{int}$. (18)

Case II. HPM

The HPM is a robust mathematical tool that introduces a homotopy parameter into an initial approximation, facilitating the systematic perturbation of the solution in nonlinear differential equations. By continuously deforming from a known solution to the desired one, and iteratively the approximation through perturbation theory. HPM excels in solving complex nonlinear problems. Its strength lies in the seamless transition from simpler to more intricate solutions, making it a versatile and effective method for tackling challenging mathematical and scientific conundrums.

To solve (12), we create the subsequent homotopy as follows;

$\mathbb{V}_{tt} - \mathbb{U}_{0tt} + \mathbb{P}(\mathbb{U}_{0tt} - \mathbb{V}_{xx} - 2\mathbb{V} + 2\mathbb{S}\text{inx}\mathbb{S}\text{int}) = 0$, (19)

Consider the solution of equation in the form as;

$\mathbb{V} = \mathbb{V}_0 + \mathbb{P}\mathbb{V}_1 + \mathbb{P}^2\mathbb{V}_2 + \mathbb{P}^3\mathbb{V}_3 + \cdots$ (20)

We obtain a system of linear equations by substituting equation (19) in equation (20) and equating coefficient of like powers $\mathbb{P}$.

$\mathbb{P}^0 = \mathbb{V}_{0tt} - \mathbb{U}_{0tt} = 0$,

$\mathbb{P}^1 = \mathbb{V}_{1tt} + \mathbb{U}_{0tt} - \mathbb{V}_{0xx} - 2\mathbb{V}_0$

$= -2\mathbb{S}\text{inx}\mathbb{S}\text{int}, \mathbb{V}_1(x,0) = 0, \mathbb{V}_{1t}(x,0) = 0$,

$\mathbb{P}^2 = \mathbb{V}_{2tt} - \mathbb{V}_{1xx} - 2\mathbb{V}_1 = 0, \mathbb{V}_2(x,0) = 0, \mathbb{V}_{2t}(x,0) = 0$,

$\mathbb{P}^3 = \mathbb{V}_{3tt} - \mathbb{V}_{2xx} - 2\mathbb{V}_2 = 0, \mathbb{V}_3(x,0) = 0, \mathbb{V}_{3t}(x,0) = 0$, Setting $\mathbb{V}_0 = \mathbb{U}_0 = \mathbb{t}\mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}$ thus solving above system we get,

$\mathbb{V}_0 = \mathbb{t}\mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}$,

$\mathbb{V}_1(\mathbb{x},\mathbb{t}) = \mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}\left(\frac{\mathbb{t}^3}{3!} + 2\mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}\right) - 2\mathbb{t}\mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}$,

$\mathbb{V}_2(\mathbb{x},\mathbb{t}) = \frac{1}{6}\mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}\left(\frac{\mathbb{t}^5}{20} - 12\mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x} - 2\mathbb{t}^3\right) + 2\mathbb{t}\mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}$, (21)

$\mathbb{V}_3(\mathbb{x},\mathbb{t}) = \frac{1}{20}\mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}\left(\frac{\mathbb{t}^7}{42} + 240\mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x} - 2\mathbb{t}^5 + 40\mathbb{t}^3\right) - 2\mathbb{t}\mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}$,

You may write the approximate series solution as follows;

$\mathbb{U} = \mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}\left(\mathbb{t} - \frac{\mathbb{t}^3}{3!} + \frac{\mathbb{t}^5}{5!} - \cdots\right)$. (22)

Additionally, this will produce the closed form solution in the limit of an unlimited number of terms.

$\mathbb{U}(\mathbb{x},\mathbb{t}) = \mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}\mathbb{S}\mathbb{i}\mathbb{m}\mathbb{t}$. (23)

Example 3.2

Think about homogeneous linear Klein-Gordan equation

$\mathbb{U}_{\mathbb{t}\mathbb{t}} - \mathbb{U}_{\mathbb{x}\mathbb{x}} = \mathbb{U}$, (24)

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$\mathbb{U}(\mathbb{X},0) = 1 + \mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}, , \frac{\partial \mathbb{U}}{\partial \mathbb{t}}(\mathbb{x},0) = \mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}$. (25)

Case I. VIM

The VIM is versatile mathematical approach for solving differential equations, systematically refining initial approximations through the construction of correctional functional and incorporating an auxiliary parameter accuracy in solutions.

The correction functional is given by

$\mathbb{U}_{\mathbb{m}+1}(\mathbb{x},\mathbb{t}) = \mathbb{U}_{\mathbb{m}}(\mathbb{x},\mathbb{t}) + \int_0^{\mathbb{t}} \lambda(\xi)\left(\frac{\partial^2 \mathbb{U}_{\mathbb{m}}(\mathbb{x},\xi)}{\partial \xi^2} - \frac{\partial^2 \mathbb{U}_{\mathbb{m}}(\mathbb{x},\xi)}{\partial \mathbb{x}^2} - \mathbb{U}_{\mathbb{m}}(\mathbb{x},\xi)\right) d\xi$, (26)

Finding the lagrange multiplier after making the correction functionally stationary can be done as follows:

$\lambda = (\xi - \mathbb{t})$.

$\mathbb{U}_{\mathbb{m}+1}(\mathbb{x},\mathbb{t}) = \mathbb{U}_{\mathbb{m}}(\mathbb{x},\mathbb{t}) + \int_0^{\mathbb{t}} (\xi - \mathbb{t})\left(\frac{\partial^2 \mathbb{U}_{\mathbb{m}}(\mathbb{x},\xi)}{\partial \xi^2} - \frac{\partial^2 \mathbb{U}_{\mathbb{m}}(\mathbb{x},\xi)}{\partial \mathbb{x}^2} - \mathbb{U}_{\mathbb{m}}(\mathbb{x},\xi)\right) d\xi$, (27)

Using the selection $\mathbb{U}_0(\mathbb{x},\mathbb{t}) = 1 + \mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}$ provides the iterative approximations as

$\mathbb{U}_1(\mathbb{x},\mathbb{t}) = \mathbb{U}_0(\mathbb{x},\mathbb{t}) + \int_0^{\mathbb{t}} (\xi - \mathbb{t})\left(\frac{\partial^2 \mathbb{U}_0(\mathbb{x},\xi)}{\partial \xi^2} - \frac{\partial^2 \mathbb{U}_0(\mathbb{x},\xi)}{\partial \mathbb{x}^2} - \mathbb{U}_0(\mathbb{x},\xi)\right) d\xi$,

$= 1 + \mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x} + \int_0^{\mathbb{t}} (\xi - \mathbb{t})(\mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x} - (1 + \mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}))\, d\xi$,

$= \frac{\mathbb{t}^2}{2!}$,

$\mathbb{U}_2(\mathbb{x},\mathbb{t}) = \mathbb{U}_1(\mathbb{x},\mathbb{t}) + \int_0^{\mathbb{t}} (\xi - \mathbb{t})\left(\frac{\partial^2 \mathbb{U}_1(\mathbb{x},\xi)}{\partial \xi^2} - \frac{\partial^2 \mathbb{U}_1(\mathbb{x},\xi)}{\partial \mathbb{x}^2} - \mathbb{U}_1(\mathbb{x},\xi)\right) d\xi$, (28)

$= \frac{\mathbb{t}^4}{4!}$,

$\mathbb{U}_3(\mathbb{x},\mathbb{t}) = \frac{t^6}{6!}$,

The series solution is provided by

$\mathbb{U}(\mathbb{x},\mathbb{t}) = \mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x} + 1 + \frac{\mathbb{t}^2}{2!} + \frac{\mathbb{t}^4}{4!} + \frac{\mathbb{t}^6}{6!} + \frac{\mathbb{t}^8}{8!} + \cdots$, (29)

And the following is the closed form solution:

$\mathbb{U}(\mathbb{x},\mathbb{t}) = \mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x} + \mathbb{C}\mathbb{o}\mathbb{s}\mathbb{t}$. (30)

Case II. HPM

The HPM is a powerful mathematical technique that introduces a homotopy parameter into initial approximations, facilitating systematic perturbation of solutions in nonlinear differential equations, providing a versatile tool for solving complex problems. According to homotopy we have;

Beginning with $\mathbb{V}_0 = 1 + \mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x}$ and from the recreation formula we have;

$\mathbb{V}_1 = \int_0^{\mathbb{t}}\int_0^{\mathbb{t}} (\mathbb{V}_{0\mathbb{x}\mathbb{x}} + \mathbb{V}_0 - \mathbb{U}_{0\mathbb{t}\mathbb{t}}) d\mathbb{t} d\mathbb{t} = \frac{\mathbb{t}^2}{2}$,

$\mathbb{V}_2 = \int_0^{\mathbb{t}}\int_0^{\mathbb{t}} (\mathbb{V}_{1\mathbb{x}\mathbb{x}} + \mathbb{V}_1) d\mathbb{t} d\mathbb{t} = \frac{\mathbb{t}^4}{24}$,

$\mathbb{V}_3 = \int_0^{\mathbb{t}}\int_0^{\mathbb{t}} (\mathbb{V}_{2\mathbb{x}\mathbb{x}} + \mathbb{V}_2) d\mathbb{t} d\mathbb{t} = \frac{\mathbb{t}^6}{720}$,

$\vdots\ \vdots\ \vdots$ (32)

Hence, the answer to the approximation series is,

$\mathbb{U}(\mathbb{x},\mathbb{t}) = \mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x} + 1 + \frac{\mathbb{t}^2}{2} + \frac{\mathbb{t}^4}{24} + \frac{\mathbb{t}^6}{720} + \cdots$, (33)

The closed-from solution in the limit of an infinite number of terms will follow from this $\mathbb{U}(\mathbb{x},\mathbb{t}) = \mathbb{S}\mathbb{i}\mathbb{m}\mathbb{x} + \mathbb{C}\mathbb{o}\mathbb{s}\mathbb{t}$.. (34)

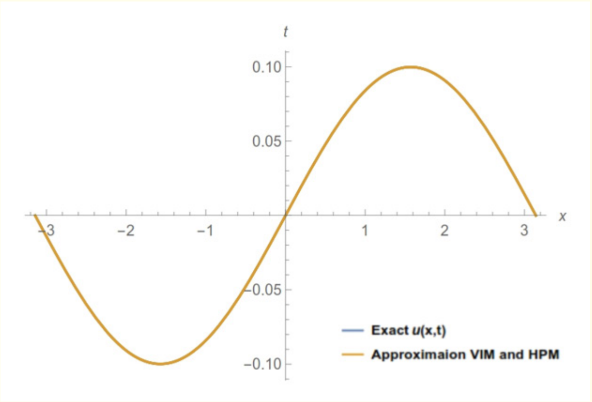

Figure 1: Approximate solution of equation (12) when x = 0: π, t = 0.1.

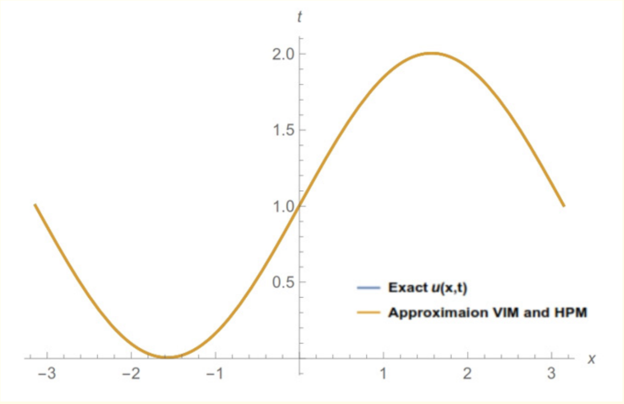

Figure 2: Approximate solution of equation (24) when when x = 0: π, t = 0.1.

	Approximation VIM and HPM	Exact
0	0	0
$\frac{\pi}{6}$	0.04991670833333334	0.04991670832341408
$\frac{\pi}{3}$	0.08645827497993011	0.08645827496274945
$\frac{\pi}{2}$	0.09983341666666667	0.09983341664682815
$\frac{2\pi}{3}$	0.08645827497993011	0.08645827496274945
$\frac{5\pi}{6}$	0.04991670833333334	0.04991670832341408
π	0	0

Table 1: Numerical value of the equation (12).

	Approximation VIM and HPM	Exact
0	1.0050041680555555	1.0050041680558035
$\frac{\pi}{6}$	1.5050041680555555	1.5050041680558035
$\frac{\pi}{3}$	1.871029571839994	1.871029571840242
$\frac{\pi}{2}$	2.0050041680555557	2.0050041680558035
$\frac{2\pi}{3}$	1.871029571839994	1.871029571840242
$\frac{5\pi}{6}$	1.5050041680555555	1.5050041680558035
π	1.0050041680555555	1.0050041680558035

Table 2: Numerical value of the equation (24).

Conclusion

In this paper, this paper presents a novel and direct approach to solving Klein-Gordon equations (KGEs) using the VIM and the HPM. Notably, our methodology eschews the need for linearization, transformation, perturbation, discretization, or confining presumptions, highlighting the simplicity and efficacy of the suggested techniques. The distinct advantage of our approach lies in its ability to solve problems without relying on Adomian's polynomials, setting it apart from the decomposition method. The VIM, employed in this study, systematically refines initial approximations by constructing a correctional functional and introducing an auxiliary parameter. This process allows for the iterative improvement of solutions, leading to accurate and meaningful results. The versatility of VIM is particularly advantageous, as it can be used in a variety of range of differential equations, providing a systematic and efficient means of tackling both linear and nonlinear problems. Similarly, the HPM proves to be a powerful tool in solving Klein-Gordon equations. By introducing a homotopy parameter into the initial approximation and employing perturbation theory, HPM facilitates the systematic perturbation of solutions in nonlinear differential equations. This method's unique ability to smoothly transition from known to desired solutions makes it well-suited for addressing complex nonlinear problems, adding to its versatility and effectiveness in mathematical modeling and scientific research. The direct application of these methods to Klein-Gordon equations showcases their robustness and reliability in capturing the intricate dynamics of scalar fields. The absence of the need for Adomian's polynomials, a distinctive feature of our approach, contributes to its computational efficiency and simplicity. The solutions obtained through VIM and HPM are not only accurate but also emphasize the practical applicability of these methods in solving real-world problems in diverse scientific disciplines. As we advance in our understanding of mathematical techniques for solving differential equations, the significance of methods like VIM and HPM becomes increasingly apparent. Their ability to address complex problems without resorting to cumbersome mathematical procedures positions them as valuable tools in the toolkit of researchers and practitioners. This study contributes to this evolving landscape by demonstrating the effectiveness of VIM and HPM in the context of Klein-Gordon equations, encouraging further exploration and application in other challenging mathematical and physical scenarios.

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