Acta Scientific COMPUTER SCIENCES

Volume 6 Issue 3 March 2024

Research Article

Some Properties and Applications of the New Integral Transform - Chinchole Transform

Ahmed Bokhari and Rachid Belgacem*

Department of Mathematics, Faculty of Exact Sciences and Informatics, Hassiba and Benbouali University of Chlef, Algeria

***Corresponding Author:** Rachid Belgacem, Department of Mathematics, Faculty of Exact Sciences and Informatics, Hassiba and Benbouali University of Chlef, Algeria.

Received: February 08, 2024

Published: February 23, 2024

Abstract

In this work, we discuss some of the properties of an integral transformation that was presented recently. This transformation is considered a generalization of the Laplace transform, so that the exponential function in the Laplace transform was replaced by the two parameters Mittag-Leffler function, which makes its study somewhat complicated. The properties discussed are used to solve some types of differential equations.

Keywords: Chinchole Transform; Laplace Transform; Mittag-Leffler Function; Linearity Property; Scale Property

Introduction

Integral transformations are mathematical operations that involve transforming a function Integral transformations are mathematical operations that involve transforming a function or equation from one domain to another using integration. These transformations are widely used in various branches of mathematics, physics, engineering, and other scientific disciplines.

There are several important integral transformations, each with its own specific properties and applications. Here are some commonly used integral transformations: Laplace Transform: The Laplace transform is a widely used integral transformation that converts a function of time into a function of complex frequency. It is particularly useful in solving linear ordinary and partial differential equations.

The Laplace transform of a function f(t) is denoted by F(s) and is defined as [3,9,12]:

$$F(s) = \int_0^{\infty} f(t)\, e^{-st} dt,$$

Where s is a complex variable.

Fourier Transform: The Fourier transform is a mathematical technique that decomposes a function into its constituent frequencies. It converts a function of time (or space) into a function of frequency.

The Fourier transform of a function f(t) is denoted by F(w) and is defined as [1,10,11]:

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-iwt} dt,$$

Where is the angular frequency.

Mellin Transform: The Mellin transform is an integral transformation that generalizes the Laplace transform and the Fourier transform. It transforms a function into a function of a complex variable raised to a power. The Mellin transform of a function f(t) is denoted by M(s) and is defined as [7,12]:

$$M(s) = \int_0^{\infty} t^{s-1} f(t)\, dt,$$

Where is a complex variable.

These are just a few examples of integral transformations, and there are many others, each with its own applications and properties. Integral transformations provide powerful tools for solving various mathematical and physical problems by converting them into alternative domains where they might be more easily analyzed or solved.

In this work, we discuss some of the properties of an integral transformation that was presented recently by Chinchole in [1]. This transformation is considered a generalization of the Laplace transform, so that the exponential function in the Laplace transform was replaced by the two parameters Mittag-Leffler function, which makes its study somewhat complicated.

Preliminaries

Definition 2.1 [5,6]. The function Mittag-Leffler which is denoted by E_σ (t) is given by

$$E_\sigma(t)=\sum_{j=0}^{\infty}\frac{t^j}{\Gamma(j\sigma+1)},\qquad \sigma>0.$$

For σ = 1 we have E_1 (t) = exp(t).

Definition 2.2 [2]. The two-parametric Mittag-Leffler function is defined by

$$E_{\sigma,\mu}(t)=\sum_{j=0}^{\infty}\frac{t^j}{\Gamma(j\sigma+\mu)},\qquad \sigma>0,\mu>0.$$

For μ = 1, we have $E_{\sigma,1}$ (t) = E_σ (t).

Definition 2.3 [2]. The Chinchole transform is obtained over the set of functions
$H=\{\nu(t)\in PC(]0,\infty[);\exists M>0,|\nu(t)|<Me^{-kt}$ for some $k>0.\}$

By

$$C_{\sigma,\mu}\{\nu(t);s\}=\int_0^\infty t^{\mu-1}E_{\sigma,\mu}(-st^\sigma)\nu(t)\,dt,\quad s>0.$$

(2.1)

Some results on Chinchole transform

Linearity Property of Chinchole transform

Let the functions f(t) and g(t) be in set , then the function af(t) + bg(t) in , where a and b are nonzero arbitrary constants, and $C_{\sigma,\mu}$ {af(t) + bg(t); s} = a$C_{\sigma,\mu}$ {f(t);s} + b$C_{\sigma,\mu}$ {g(t);s}. (3.1)

Proof. Using the Definition 2 of Chinchole transform, we get

$$\begin{aligned} C_{\sigma,\mu}\{af(t)+bg(t);s\} &= \\ &= \int_0^\infty t^{\mu-1}E_{\sigma,\mu}(-st^\sigma)(af(t)+bg(t))\,dt \\ &= \int_0^\infty t^{\mu-1}E_{\sigma,\mu}(-st^\sigma)(af(t))\,dt+\int_0^\infty t^{\mu-1}E_{\sigma,\mu}(-st^\sigma)(bg(t))\,dt, \\ &= aC_{\sigma,\mu}\{f(t);s\}+bC_{\sigma,\mu}\{g(t);s\}. \end{aligned}$$

Change of scale property of Chinchole transform

Proposition 3.1 If the function belongs to set , then

$$C_{\sigma,\mu}\{f(at);s\}=\frac{1}{a^\mu}C_{\sigma,\mu}\left\{f(t);\frac{s}{a^\sigma}\right\}. \quad (3.2)$$

Proof. Using the Definition 2 of Chinchole transform, we get

$$C_{\sigma,\mu}\{f(at);s\}=\int_0^\infty t^{\mu-1}E_{\sigma,\mu}(-st^\sigma)f(at)\,dt. \quad (3.3)$$

Substituting ζ = at in Equ.(3.3) yields

$$\begin{aligned} C_{\sigma,\mu}\{f(at);s\} &= \int_0^\infty\left(\frac{\zeta}{a}\right)^{\mu-1}E_{\sigma,\mu}\left(-s\left(\frac{\zeta}{a}\right)^\sigma\right)f(\zeta)\frac{d\zeta}{a} \\ &= \frac{1}{a^\mu}\int_0^\infty\zeta^{\mu-1}E_{\sigma,\mu}\left(\frac{-s}{a^\sigma}\zeta^\sigma\right)f(\zeta)\,d\zeta \\ &= \frac{1}{a^\mu}C_{\sigma,\mu}\left\{f(t);\frac{s}{a^\sigma}\right\}. \end{aligned}$$

Example 3.1 Since

$$C_{1,2}\{e^{-t};s\}=\frac{1}{1+s}, \quad (3.4)$$

Then we have

$$C_{1,2}\{e^{-at};s\}=\frac{1}{a^2}C_{1,2}\left\{e^{-t};\frac{s}{a^1}\right\}=\frac{1}{a^2}\frac{1}{1+\frac{s}{a}}=\frac{1}{a(a+s)},$$

(3.5)

$$\begin{aligned} C_{\sigma,\mu}\{f(rt);s\} &= \frac{1}{r^\mu}C_{\sigma,\mu}\left\{f(t);\frac{s}{r}\right\}=\frac{1}{r^\mu}\frac{a^{\sigma-\mu}}{a^\sigma+\frac{s}{r}} \\ &= \frac{a^{\sigma-\mu}}{r^{\mu-1}(ra^\sigma+s)} \quad (3.6) \end{aligned}$$

$$\begin{aligned} C_{1,2}\left\{\frac{1}{(1+t)^2};s\right\} &= \int_0^\infty tE_{1,2}(-st)\frac{1}{(1+t)^2}dt \\ &= -\frac{1}{s}\int_0^\infty\frac{e^{-st}-1}{(1+t)^2}dt \\ &= \frac{1-\mathbf{E}_2(s)e^s}{s} \end{aligned}$$

Where is the MacRobert's E-Function [8].

$$\begin{aligned} C_{1,2}\left\{\frac{1}{(1+at)^2};s\right\} &= \frac{1}{a^2}C_{1,2}\left\{\frac{1}{(1+t)^2};\frac{s}{a}\right\} \\ &= a\frac{1-\mathbf{E}_2\left(\frac{s}{a}\right)e^{\frac{s}{a}}}{s}. \end{aligned}$$

Proposition 3.2 Let be the derivative of such as Then

$$C_{\sigma,1}\{u^{(n)}(t);s\}=-u^{(n-1)}(0)+sC_{\sigma,\sigma}\{u^{(n-1)}(t);s\}.$$

--------(3.7)

Proof. We use here the rule of integration by parts, as given in [1]. By definition, we have

$$C_{\sigma,1}\{u^{(n)}(t);s\}=\int_0^\infty E_{\sigma,1}(-st^\sigma)u^{(n)}(t)dt,$$

Then

$$\begin{aligned} C_{\sigma,1}\{u^{(n)}(t);s\} &= \\ &= \int_0^\infty E_{\sigma,1}(-st^\sigma)u^{(n)}(t)dt \\ &= \left[E_{\sigma,1}(-st^\sigma)u^{(n)}(t)\right]_0^\infty-\int_0^\infty\left(-st^{\sigma-1}\right)E_{\sigma,\sigma}(-st^\sigma)u^{(n-1)}(t)dt \\ &= \left[E_{\sigma,1}(-st^\sigma)u^{(n)}(t)\right]_0^\infty+s\int_0^\infty t^{\sigma-1}E_{\sigma,\sigma}(-st^\sigma)u^{(n-1)}(t)dt \\ &= -u^{(n-1)}(0)+sC_{\sigma,\sigma}\{u^{(n-1)}(t);s\}. \end{aligned}$$

Corollary 3.1 If σ = 1, then the Chinchole transform is identical to the Laplace transform, thus

$$C_{1,1}\left\{u^{(n)}(t);s\right\}=L\left\{u^{(n)}(t);s\right\}$$
$$=s^nL\left\{u(t);s\right\}-\sum_{k=0}^{n-1}s^{n-k-1}u^{(k)}(0). \quad (3.8)$$

If σ > 1, then

$$C_{\sigma,\sigma}\left\{u^{(n)}(t);s\right\}=-C_{\sigma,\sigma-1}\left\{u^{(n-1)}(t);s\right\}. \quad (3.9)$$

For σ = 2, we have

$$C_{2,2}\left\{u^{(n)}(t);s\right\}=u^{(n-2)}(0)-sC_{2,1}\left\{u^{(n-2)}(t);s\right\}$$

For σ > 2, we have

$$C_{\sigma,\sigma}\left\{u^{(n)}(t);s\right\}=-C_{\sigma,\sigma-1}\left\{u^{(n-1)}(t);s\right\}$$
$$=C_{\sigma,\sigma-2}\left\{u^{(n-2)}(t);s\right\}, \quad (3.10)$$

And for α > j

$$C_{\sigma,\sigma}\left\{u^{(n)}(t);s\right\}=(-1)^jC_{\sigma,\sigma-j}\left\{u^{(n-j)}(t);s\right\}. \quad (3.11)$$

Proof. Using the Definition 2 of Chinchole transform, we get

$$\begin{aligned}C_{1,1}\left\{u(t);s\right\} &= \int_0^\infty E_{1,1}(-st)\,u(t)\,dt\\ &= \int_0^\infty \exp(-st)\,u(t)\,dt\\ &= L\left\{u(t);s\right\}.\end{aligned}$$

Therefore, the relation (3.8) is satisfied. If , then

$$0 \le \left|t^{\sigma-1}E_{\sigma,\sigma}(-st^\sigma)\,u^{(n-1)}(t)\right| \le \left|t^{\sigma-1}E_{1,1}(-st^\sigma)\,u^{(n-1)}(t)\right|$$
$$\le \left|t^{\sigma-1}\exp(-st^\sigma)\,M\exp(-kt)\right|$$

for some k > 0

$$\le M\frac{t^{\sigma-1}}{\exp(st^\sigma+kt)} \longrightarrow 0 \qquad \text{when . } t\to\infty.$$

So

$$\left[t^{\sigma-1}E_{\sigma,\sigma}(-st^\sigma)\,u^{(n-1)}(t)\right]_0^\infty=0$$

Integration by parts gives

$$\begin{aligned}C_{\sigma,\sigma}\left\{u^{(n)}(t);s\right\} &= \\ &= \int_0^\infty t^{\sigma-1}E_{\sigma,\sigma}(-st^\sigma)\,u^{(n-1)}(t)\,dt\\ &= \left[t^{\sigma-1}E_{\sigma,\sigma}(-st^\sigma)\,u^{(n-1)}(t)\right]_0^\infty-\int_0^\infty t^{\sigma-2}E_{\sigma,\sigma-1}(-st^\sigma)\,u^{(n-1)}(t)\,dt\\ &= -C_{\sigma,\sigma-1}\left\{u^{(n-1)}(t);s\right\}.\end{aligned}$$

For σ = 2, we use equations (3.9) and (3.7) to obtain

$$\begin{aligned}C_{2,2}\left\{u^{(n)}(t);s\right\} &= \\ &= -C_{2,1}\left\{u^{(n-1)}(t);s\right\} \quad (3.12)\\ &= -\left(-u^{(n-2)}(0)+sC_{2,1}\left\{u^{(n-2)}(t);s\right\}\right)\\ &= u^{(n-2)}(0)-sC_{2,1}\left\{u^{(n-2)}(t);s\right\}.\end{aligned}$$

For σ > 2, by applying the relation (3.9) twice, we get, Eq.(3.10).
For α > j , by induction, we get desired result.
Example 3.2

$$C_{\sigma,\mu}\left\{u^{(4)}(t);s\right\}=$$
$$=\begin{cases} s^4L\left\{u(t);s\right\}-\sum_{k=0}^{3}s^{3-k}u^{(k)}(0), & if\ \sigma=\mu=1\\ s^2C_{2,1}\left\{u(t);s\right\}+su\prime(0)-u^{(3)}(0), & if\ \sigma=2,\mu=1\\ s^2C_{3,3}\left\{u(t);s\right\}-su(0)-u^{(3)}(0), & if\ \sigma=3,\mu=1\\ -sC_{\sigma,\sigma-3}\left\{u(t);s\right\}-u^{(3)}(0), & if\ \sigma>3,\mu=1\\ C_{\sigma,\mu-4}\left\{u(t);s\right\}, & if\ \mu>4.\end{cases}$$

$$C_{\sigma,\mu}\left\{u^{(5)}(t);s\right\}=$$
$$=\begin{cases} s^5L\left\{u(t);s\right\}-\sum_{k=0}^{4}s^{4-k}u^{(k)}(0), & if\ \sigma=\mu=1\\ s^3C_{2,2}\left\{u(t);s\right\}-s^2u(0)+su''(0)-u^{(4)}(0), & if\ \sigma=2,\mu=1\\ -s^2C_{3,1}\left\{u(t);s\right\}-su'(0)-u^{(4)}(0), & if\ \sigma=3,\mu=1\\ -s^2C_{4,4}\left\{u(t);s\right\}+su(0)-u^{(4)}(0), & if\ \sigma=4,\mu=1\\ sC_{\sigma,\sigma-4}\left\{u(t);s\right\}-u^{(4)}(0), & if\ \sigma>4,\mu=1\\ -C_{\sigma,\mu-5}\left\{u(t);s\right\}, & if\ \mu>5.\end{cases}$$

***f* (*t*)**	**$C_{\sigma,\mu}$ {*f* (*t*);*s*}**	
e−*t*	1 1+*s*	any σ,μ
e−*at*	$\frac{a^{\sigma-\mu}}{a^{\sigma}+s}$	any σ,μ
tn	$i\frac{\Gamma(n+1)}{s^{n+\frac{3}{2}}}$	σ = 1, *μ* = 2
$\frac{1}{(1+at)^2}$	$a\frac{1-\mathbf{E}_2\left(\frac{s}{a}\right)e^{\frac{s}{a}}}{s}$	σ = 1, *μ* = 2
cos(*kt*)	$\frac{\pi}{2\sqrt{s}}\delta\left(\frac{k}{\sqrt{s}}-1\right)$	σ = 1, *μ* = 2

Table 1: Chinchole transform of some functions.

Conclusion

In this work, we reviewed some of the properties of the modern integral transformation, the Chinchole transformation, such as linearity and some relationships between derivatives and their transformations. In the upcoming works, we seek to prove other properties of this transformation and search, especially for applications in practical life.

The presence of the Mittag-Leffler function in the kernel of this transformation gives an indication of its connection to some fractional derivatives, such as the Atangana–Baleanu operator and the Prabhakar operator. Discovering this type of connection may provide many facilities in studying the issues related to the aforementioned operators.

Acknowledgements

The authors would like to thank the General Directorate of Scientific Research and Technological Development (DGRS-DT) of Algeria for supporting partially the research project N° C00L03UN020120220002.

Bibliography

1. RN Bracewell. "The Fourier Transform and Its Applications". 3rd ed. McGraw-Hill. New York, (2000).
2. SM Chinchole. "A New Integral Transform With Applications to Fractional Calculus". *Communications in Mathematics and Applications* 13.3 (2022): 1003 1012.
3. L Debnath and D Bhatta, "Integral transforms and their applications". Second edition, Chapman and Hall/CRC, (2006).
4. A Erdelyi. "Higher Transcendental Functions". McGraw-Hill, New York, 3 (1955).
5. P Humbert. "Quelques résultats relatifs la fonction de Mittag-Leffler". C. R. Acad. Sci. Paris 236 (1953): 1467-1468.
6. GM Mittag- Leffler. "Sur la nouvellc fonction Eα(x)". C. R. Aca.d. Set. Pans 137 (1903): 554-558.
7. F Oberhettinger. " Tables of Mellin Transforms". Springer, New York (1974).
8. FM Ragab. "MacRobert's E-Functions". University of Glasgow (United Kingdom) ProQuest Dissertations Publishing, (1953).
9. JL Schi . "The Laplace Transform: Theory and Application". New York: Springer (1999).
10. IN Sneddon. "Fourier Transforms". Dover, New York (1995). [First published by McGraw-Hill in 1951].
11. EC Titchmarsh. "Introduction to the Theory of Fourier Integrals". Oxford Univ. Press, Oxford, (1948).
12. DV Widder. "An introduction to transform theory". Pure Appl. Math., Academic Press, New York, 42 (1971).