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Review Article

Manifolds and Lie groups

UMSALAMA Ahmed Abd ALLA ElEMAM[1,2]

[1]Department of Mathematics, Faculty of Addayer College University, Saudi Arabia

[2]Department of Mathematics, Faculty of Education Jazan University, Saudi Arabia

***Corresponding Author:** UMSALAMA Ahmed Abd ALLA ElEMAM, Department of Mathematics, Faculty of Addayer College University, Saudi Arabia.

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Abstract

In this study, we introduce a manifold which help to give further understanding of spaces and applied features of spaces. we have introduced Riemannian Manifold, Lie groups. In mathematics, a manifold is a topological space that locally resembles Euclidean space near each point. More precisely, an -dimensional manifold, or -manifold for short, is a topological space with the property that each point has a neighborhood that is homeomorphic to an open subset of -dimensional Euclidean space. One-dimensional manifolds include lines and circles, but not self-crossing curve. concept of a manifold is central to many parts of geometry and modern mathematical physics because it allows complicated structures to be described in terms of well-understood topological properties of simpler spaces. Manifolds naturally arise as solution sets of systems of equations and as graphs of functions.

Keywords: The R^n Space; Euclidean Space; Homeomorphism; Housdorff Space; Metric Spaces; Topological Manifold Differentiable Manifolds; Sub Manifolds; Riemannian Manifold; Vector Field; Lie Groups

The R^n space

In R^n space the neighborhood are then open balls $B^n_\in(x)$ or $B_\in(x)$ or equivalently open cubes $C^n_\in(x)$ defined for any $\in > 0$ as:

$\{B_\in(x) = y \in R^n | x^i - y^i < \in , i = 1, \dots, n\}$. R^n may denote a vector space called Euclidean if it has been defined on it a positive definite inner product $(x, y) = \sum_{i=1}^n x^i y^i$, thus R^n is Euclidean vector space but one which has a built in orthonormal basis and inner product. We define the norm $\|x\|$ of the vector X by $\|x\| = ((x, x))^{\frac{1}{2}}$ then we have: $d(x, y) = \|x - y\|$

The notation is frequently useful even when we are dealing with R^n as a metric space and not using its vector space structure

Definition (Euclidean space)

Euclidean n-space, sometimes called Cartesian space or simply $n -$space, is the set of all n-tuples of real numbers $(x_1 , x_2 , \dots \dots , x_n)$ its commonly denoted by R^n.

Euclidean $3 -$space R^3 is the set of all ordered triples of real numbers. Such a triple $P = (p_1, p_2, p_3)$ is called a point of R^3.

Euclidean $2 -$ space R^2 is set of all ordered pairs of real numbers and also called real plane.

$R^1 = R$ is the set of all real numbers (i.e the real n-line).

Definition (locally Euclidean)

A topological space X is called locally Euclidean if there is a $non -$negative integer n such that every point in X has a neighborhood which is homeomorphic to an open subset of Euclidean space R^n.

Example

Any open subset $X \subset R^n$ is a locally Euclidean space that is also Hausdorff and 2nd countable.

Definition (Homeomorphism)

Let X , Y be topological spaces. A function $f : X \rightarrow Y$ is said to be a homeomorphism if and only if:

- f is a bijective.
- f is continuous over X and (III) f^{-1} is continuous over.

If a homeomorphism is exists between two spaces are said to be homeomorphic or topologically equivalent and we denote by $X \cong Y$.

Examples

- Any open interval in R is homeomorphism to R.

- The unit2-disc D^2 and the unit square in R^2 are homeomorphism.
- R^m *and* R^n are not homeomorphic for $m \neq n$.

Definition (local homeomorphism)

Let X and Y be topological spaces. A function $f : X \to Y$ is local home- morphism if for every point x in X, there exist an open set U containing x, such that $f(u)$ is open in Y and $f|_{u} : u \to f(U)$ is a homeomorphism.

Example

If U is an open subset of y equipped with subspace topology then the inclusion map $i : V \to Y$ is a local homomorphism.

Definition (Housdorff space)

A topological space X is said to be a T_2 - space, also called a Housdorff space, if and only if for any two distinct point x and y of X there exist distinct open subset U and V of X such that $x \in U$ *and* , $g \in V$ (See figure 1).

Figure 1

Definition (Second countable space)

Let (X, τ) be a topological space, then X is said to be second countable space, if τ has a countable basis. In other words, a topological space (X, τ) its said to be second countable space if has a countable open base.

Example

If X is finite, then member of each τ on X is finite. So its base is finite. Hence (X, τ) is second countable space.

Topological manifold

Euclidean space and their subspace R^n are the most important. The metric space R^n serve as a topological model for Euclidean spaceE^n, for finite dimensional vector spaces over R or C. It is natural enough that we are led to study those spaces which are locally like R^n.

We will consider spaces called manifolds, defined as follows.

Definition

A manifold M of dimension n, or n-manifold is topological space with the following properties :

- M is Housdorff space.
- M is locally Euclidean of dimension n and, (III) M has a countable basis of open sets.

As a matter of notion dim M is used for the dimension of M, when dim$=0$, then M is a countable space with discrete topology.

Examples

(Circle). Define the circle $S^1 = \{z \in \mathbb{C}: |z| = 1\}$. Then for any fixed point $z \in S^1$,write it as $z = e^{2\pi ic}$ for a unique real number $0 \leq c \leq 1$, and define the map $v_z: t \to e^{2\pi it}$.

We note that v_z maps the natural $I_c = (c - \frac{1}{2}, c + \frac{1}{2})$ to the neighborhood of z given by $S^1/-z$, and it is a homeomorphism. Then $\varphi_z = v_z|_{I_c}^{-1}$ is a local coordinate chart near.

By taking products of coordinate charts, we obtain charts for the Cartesian product of manifolds. Hence the Cartesian product is a manifold.

(Open subsets). Any open subset $U \subset M$ of a topological manifold is also a topological manifold, where the charts are simply restrictions φ/U of charts φ for M.

For example, the real $n \times n$ matrix $Mat(n, R)$ from a vector space isomorphic to R^{n^2}, and contain an open subset
$GL(n, R) = \{A \in Mat(n, R): det \neq 0\}$,

Known as the general linear group, which therefore froms a topological manifold.

The n –torus
$T^n = S^1 \times S^1 \times \ldots S^1$ is an $n-$dimensional manifold.

If N is an n-manifold and M is an m-manifold then the product $N \times M$ is an $(n+m)$-manifold.

Theorem [12]

A topological manifold M is locally connected, locally compact, and a union of a countable collection of compact subsets; furthermore, it is normal and metrizable.

Further examples of manifolds Cutting and pasting

A hemispherical cap (including the equator) or a right circular cylinder (including the circles at the ends) is typical examples of manifold with boundary.

Definition

An $n-$dimensional manifold with boundary is a topological space X in which each point $x \in X$ has a neighborhood U_x homeomorphic to either R^n or H^n – the closed upper half – space in R^n. Where we shall mean by H^n the subspace or R^n defined by
$H^n = \{(x^1, \ldots, x^n) \in R^n \, |x^n, n \geq 0\}$.

Every point $x \in H^n$ has a neighborhood U which is homeomorphic to an open subset $\overline{U}$ called the boundary of H^n and denoted by ∂H^n (Figure 2).

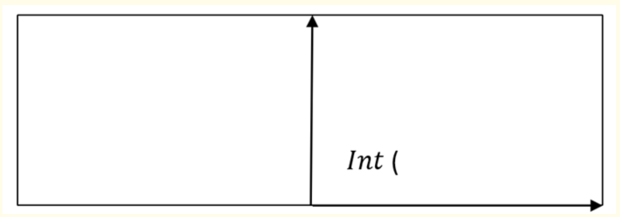

Figure 2: Upper half space.

Examples

- The upper half space H^n itself as a manifold with boundary.
- Any closed interval in R.
- Any closed disk in R^2. See also figure (3).

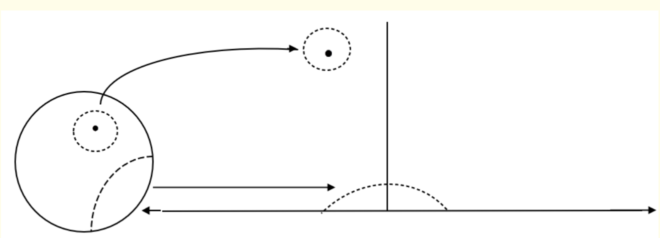

Figure 3

A manifold with boundary

- The Mobius band is a 2-dimentional manifold with boundary the circle.
- The cylinder $1\times s^1$ is a 2- dimensional manifold with boundary the union of two circles – a manifold with two components.

Our interest is in pointing out that new surfaces, that are $2-$manifolds, can be formed by fastening together manifolds with boundary along their boundaries that is by identifying points of various boundary components by a homeomorphism. The simplest examples are S^2, which is obtained by pasting two disks (or hemispheres) together so as to form the equator, and T^2, formed by pasting the two end-circles of a cylinder together. However, one can go much further and past any number of cylinders onto a sphere S^2 with circular disk removed. This gives various Pretzelike surfaces as illustrated in figure 4.

Some examples of pasting

Thus to generate new $2-$manifold from old ones we may (1) cut out two circles and (2) paste on a cylinder or "handle", so that each end – circles is identified with one of the boundary circles of M.

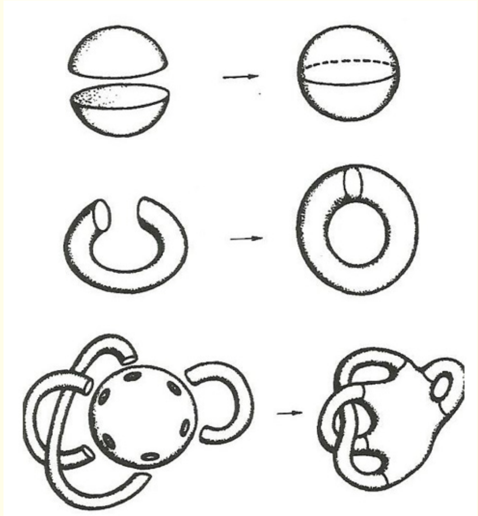

Figure 4

The pasting on of handles is not the only way in which we can generate examples of $2-$manifolds; it's also possible to do so by identifying or pasting together the edges of certain polygons. For example, the sides of square may be identifying in various ways in order to obtain surfaces (Figure 5).

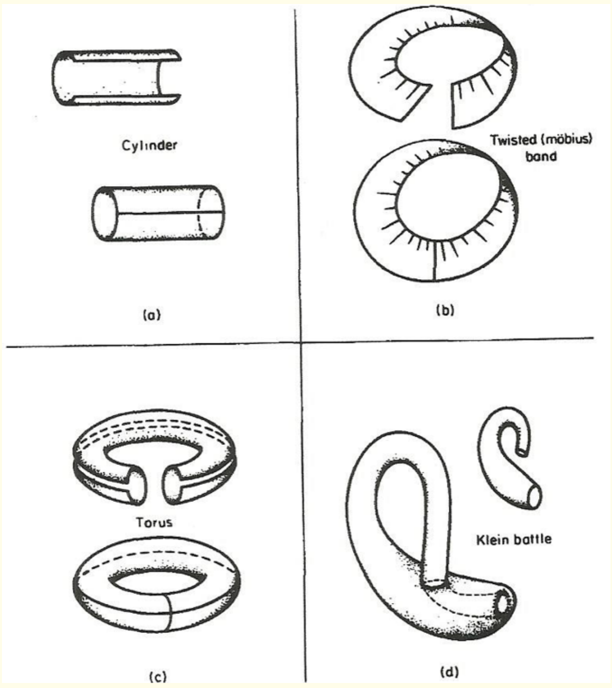

Figure 5

Four ways to identify sides of a rectangle: (a) cylinder; (b) twisted (Mobius) band (c) Tours; (d) Klein bottle.

Function of several variables and mapping

Differentiability for function of several variables:

If $f : A \to R$ is the such a function, then $f(x) = f\ (x^1, \dots, x^n)$ denotes its value at $x = (x^1, \dots, x^n) \in A$, $A \subset R$.

At each $a \in U (U \subset R^n$, and its open set) the partial derivative $(\frac{\partial f}{\partial x^j})$ of f with respect to x^j, is of course, the following limit, if it exists :

$$(\frac{\partial f}{\partial x^j})_a lim_{h\to 0} \frac{f\ (a^1, \dots, a^j + h, \dots, a^n) - f\ (a^1, \dots, a^j, \dots, a^n)}{h},$$

If $\frac{\partial f}{\partial x^j}$ is defined, that is, the limit above exists at each point of U for $1 \le j \le n$, this defines n functions on U.Should these functions be continuous on U for $1 \le j \le n$, f is said to be continuously differentiable on, denoted by $f \in C^1(U)$.

Concepts

If f is differentiable at a, then its continuous at a and all the partial derivatives $(\frac{\partial f}{\partial x^j})_a$ exist. When f is differentiable at a we have

$$\sum_{i=1}^{n} (\frac{\partial f}{\partial x^j})_a \ (x^i - a^i) + ||x - a||\ r(x, a).$$

We denote by df_a, or simply df, the homogeneous linear expression on the right :

$df = \sum_{i=1}^{n} (\frac{\partial f}{\partial x^j})_a \ (x^i - a^i)$ it's called the differential of f at $\boldsymbol{a}$.

If $\frac{\partial f}{\partial x}, \dots, \frac{\partial f}{\partial x^n}$ are defined in a neighborhood of $\boldsymbol{a}$ and continuous at $\boldsymbol{a}$, then f is differentiable at a. Thus existence and continuous at $\boldsymbol{a}$, then f is differentiable at $\boldsymbol{a}$. Thus existence and continuity of the partial derivatives of on an open set $U \subset R^n$ implies differentiability of f at every point of U.

We define inductively the notion of an r-fold continuously differentiable function on an open set $U \subset R^n$ (function of class C^r) : f is of class C^r on U if its first derivatives are of class C^{r-1}.

If f is of class C^r for all $\boldsymbol{r}$, then we say that f is smooth, or of class C^∞. We denote these class of function on U by $C^r(U)$ and $C^\infty(U)$. Define a differentiable (C^r) curve in R^n to be a mapping of an open interval $(a, b) = \{x \in R\ |a < x < b\}$ of the real number into R^n, $f : (a, b) \to R^n$ with $f(t) = \{x^1(t), \dots, x^n(t)\}$ where the coordinate function $x^1(t), \dots, x^n(t)$ are differentiable on the interval.

Now suppose that f is a differentiable curve and maps (a, b) into, an open subset of R^n.

Let $a < t_0 < b$ and suppose that g is a function on U which is differentiable at (t_0). Then the composite function $g\ o\ f$ is a real-valued function on (a, b).

W assert that $g\ o\ f$ is a differentiable at t_0 and that derivative at t_0 is given by the chain rule :

$$\frac{d}{dt}\ (g\ o\ f) = \sum_{i=1}^{n} (\frac{\partial g}{\partial x^i})_{f(t_0)}\ (\frac{dx^i}{dt})_{t_0}$$

We shall say that a domain U is star like with respect to a $\in U$ provided that whenever $x \in U$ then the segment $\overline{ax}$ lies entirely in Figure 6.

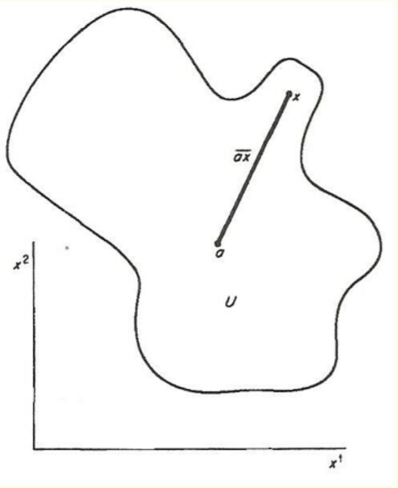

Figure 6

This is a somewhat weaker property than convexity of, a convexity set being starlike with respect to every one of its points.

Differentiability of Mapping and Jacobean

We will consider real-valued function whose domain is a subset $A \subset R^n$ and whose range is in R^m rather than R.

If $\pi^i : R^m \to R$ denotes the projection to the ith coordinate, namely $\pi^i(x^1, \dots, x^i, \dots, x^m) = x^i$, and if $f : A \to R^m$ is a mapping defined on $A \subset R^n$, then f is determined by its coordinate function $f^i = \pi^i o f$ in fact for $x \in A$. $f(x) = (f^1(x), \dots, f^m)$.

Conversely, any set of m function $f^1, \dots, f^m$ on A with values in determines a mapping $f : A \to R^m$ with the coordinates of $f(x)$ given by $(f^1(x), \dots, f^m)$ as a above.

We shall say that f is differentiable of class C^r, C^∞, C^w, and so on, at $a \in U$ or on U if each of its coordinates function has the corresponding property. We may sometimes call a C^∞ mapping F a smooth mapping; if f is smooth, then coordinate function f possesses continuous partial derivatives of all orders and each such derivative is independent of the order of differentiation.

If f is differentiable on, we know that the $m \times n$ Jacobean matrix,

$$\frac{\partial((f^1, \dots, f^m))}{\partial(x^1, \dots, x^n)} = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n} \end{pmatrix}$$

Is defined at each point of U, its mn entries being functions on U.

these functions need not be continuous on U ; they are so if and only if f is class C^1.

Definition

A mapping $: U \rightarrow R^m$, U an open subset of R^n, is differentiable at $a \in U$(or on U) if there exists an $m \times n$ matrix A of constants (respectively, function on U) and an m.tuple $R(x,a) = (r^1(x,a), \ldots, r^m(x,a))$ of function defined on U (on $U \times U$) such that $||R(x,a)|| \rightarrow 0$ as $x \rightarrow a$ and for each $x \in U$ we have :

(*) $f(x) = f(a) + A(x-a) + ||x-a||R(x,a)$.

If such $R(x,a)$ and A exist, then A is unique and is the Jacobian matrix. In above expression, A(x-a) denotes a matrix product, so we must write (x-a) as an $n \times 1$ (column) matrix and read this as an equation in $m \times 1$ matrix.

The space of tangent vectors at a point of R^n

In Euclidean space: if $\in E^3$, we let $T_a(E^3)$ be the vector space whose elements are directed line segments X_a with a as initial point. These are added by the parallelogram low: $-X_a$ is the oppositely directed segment and 0 is the segment consisting of the point a alone.

The tangent space at any two points of Euclidean space is naturally isomorphic. If a, b are points of E^3, then there is exactly one translation of the space taking a to b this translation moves each line segment issuing from a to a line segment from b and thus carries $T_a(E^3)$ to $T_b(E^3)$.

Since parallelograms go to congruent parallelograms and lengths are preserved, this correspondence is an isomorphism and its uniquely determined by the geometry (figure 7).

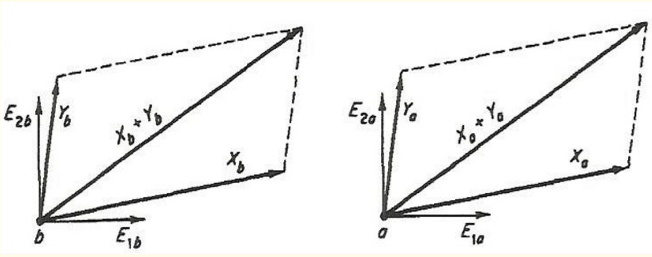

Figure 7

If we choose any a fixed point a as origin and choose at a three linearly independent vectors E_{1a}, E_{2a}, E_{3a}, for example, three naturally perpendicular unite vectors, then this will automatically determine a basis not only of $T_a(E^3)$ but also of $T_b(E^3)$ for every $b \in E^3$. Let $a = (a^1, \ldots, a^n)$ be any point of R^n. We define $T_a(R^n)$ the tangent (vectors) space attached to a, as follows.

First, as a set it consist of all pairs of point (a, x) or, $\overline{ax}$, $\boldsymbol{a} = (a^1, \ldots, a^n)$ and $x = (x^1, \ldots, x^n)$, corresponding, of course, to initial and terminal points of a segment.

We also denote such a pair by $\boldsymbol{x_a}$.

We next establish a one-to-one correspondence $\varphi_a : T_a(R^n) \rightarrow V^n$ between the set just described and the vector space of n-tuples of real numbers by the following simple device.

If $X_a = \overline{aX}$ then $\varphi_a(X_a) = (x^1 - a^1), \ldots, x^n - a^n)$.

،The vector space operation (addition and multiplication by scalars) are define in the one way possible so that φ_a is an isomorphism. This requires that The right –hand side being used to define the operation on the left. Clearly we are being guided by the fact that R^n and E^n may be identified if we choose rectangular Cartesian in E^n.

$$X_a + Y_a = \varphi_a{}^{-1}(\varphi_a(X_a) + \varphi_a(Y_a)$$

$$\alpha X_a = \varphi_a{}^{-1}(\alpha\varphi_a(X_a)), \alpha \in R^n$$

Vector field on open subsets of R^n :

A vector field on an open subset $U \subset R^n$ is a function which assigns to each point $p \in U$ a vector $X_p \in T_p(R^n)$. They are many examples in physics for $n = 2$ and $= 3$. The best known is the gravitational field : if an object of mass μ is located at a point 0, then to each point p in $U = E^n - \{0\}$ there is assigned a vector which denotes the force of attraction on a particle of unit mass placed at the point. This vector is represented by a line segment or arrow from p (as initial point) directed toward 0 and having length $K\,\mu/r^2$, $\boldsymbol{r}$ denoting the distance $d(0,p)$ and K a fixed constant determined by the unit chosen (Figure 8).

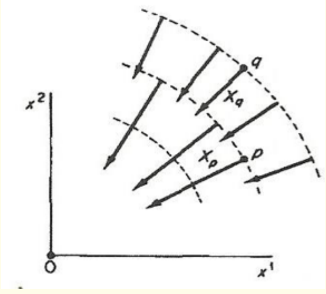

Figure 8

First quadrant portion of gravitational field of point mass at origin. If we introduce Cartesian coordinates with 0 as origin, then for the point p with coordinates (x^1, x^2, x^3) the components of X_p in the canonical basis are

$$\frac{-x^1}{r^3}, \frac{-x^2}{r^3}, \frac{-x^3}{r^3},$$

With $(x^1)^2 + (x^2)^2 + (x^3)^2)^{\frac{1}{2}}$ that is, $\frac{-1}{r^3}(x^1 E_{1p} + x^2 E_{2p} + x^3 E_{3p})$

$$= \frac{-1}{r^3}\left(x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3}\right).$$

We note that the components of X_p are C^∞ function of the co-ordinates.

We shall say that a vector field on R^n is C^∞ or smooth if its component relative to the canonical basis are C^∞ function on U.

When dealing with vector fields, as with function, the independent variable will be omitted from the notion. Thus we write X rather than X_p just as we customarily use f rather than $f(p)$ for a function.

Then X_p is the value at p of X, That is, the vector of the field which is attached to p it lies In $T_p(R^n)$.

Theorem [6]

Let $f \subset R^n$ be a closed set and $k \subset R^n$ compact, $f \cap k \neq \emptyset$ then there is a C^∞ function $\sigma(x)$ whose domain is all of R^n and whose range of value is the closed interval $[0,1]$ such that $\sigma(x) \equiv 1$von k and $\sigma(x) = 0$ on f.

The inverse function theorem

Definition (differentiable homeomorphism (diffeomorphism))

Let us suppose that $U \subset R^n$ and $v \subset R^n$ are open sets. We then shall say that a mapping $f : U \to V$ is a C^r- diffeomorphism if

- f is a homeomorphism and,
- Both f and f^{-1} are of class C^r, $r \geq 1$ (where $r = \infty$ we simply say diffeomorphism).

Its perhaps not obvious why we need to require both f and f^{-1} to be of class C^r - its because we wish the relation to be symmetric.

Example

Let $U = R$ and $V = R$ and $f : t \to s = t^3$; this is a homeomorphism.

And f is analytic but $f^{-1} : s \to t = t^{\frac{1}{3}}$ is not C^1 on V since it has no derivative at $= 0$.

Example

Let $f : R^n \to R^n$ be the translation taking a = $(a^1, \dots, a^n)$ to b = $(b^1, \dots, b^n)$. Then f is given by $f(x^1, \dots, x^n) = ((x^1 + (b^1 - a^1), \dots, x^n + (b^n - a^n))$, or $f(x) = x + (b - a)$.

The coordinate function $f^i(x) = x^i + (b^i - a^i)$ are analytic, and hence C^∞ The translation $G(x) = x + (a - b)$ *is* f^{-1} which is the then also C^∞ and since f, f^{-1} are defined and continuous f is a homeomorphism. Thus f is a diffeomorphism.

Lemma [12]

Let U, V be open subset of R^n, $f : U \to V, f : V \to W$ mappings onto, and $H = GoF : U \to W$ their composition.

If any two of these maps is a diffeomorphism, then the third is also.

Theorem [4] (inverse function theorem)

Let Wbean open subset of R^n and $f : W \to R^n$ a C^r mapping, $r = 1,2, \dots$ or ∞. If $a \in w$ and $Df(a)$ is nonsingular, then there exists an open neighborhood U of a in W such that $v = f(u)$ is open and $f: U \to V$ is a C^r diffeomorphism.

If $x \in U$ *and* $Y = f(x)$, then we have the following formula for the derivatives of f^{-1} at y = $Df^{-1}(y) = (Df(x))^{-1}$. The term on the right denoting the inverse matrix to $Df(x)$.

Corollary [1]

If Df is nonsingular at every point of W, then f is an open mapping of W, that is, it carries W and open subset of R^n contained to open subset of R^n.

The rank of a mapping

In linear algebra the rank of an $m \times n$ A is defined in three equivalent ways

- The dimension of the subspace of V^n spanned by the rows
- The dimension of the subspace of V^m spanned by the columns, and (iii) the maximum order of any nonvanishing minor determinant. We see at once from (i) and (ii) that the rank $A \leq m, n$. The rank of a linear transformation is defined to be the dimension of the image.

When $f : U \to R^m$ is a C^1 mapping of an open set $U \subset R^n$, then rank $Df(x)$ has a rank at each $x \in U$. Because the value of a determinant is a continuous function of its entries, we see from (iii) that if $Df(a) = k$, Then for some open neighborhood V of a, rank $Df(x) \geq k$; and, if $k = inf(m, n)$, then rank $Df(x) = k$ on this V. In general, the inequality is possible:

$f(x^1, x^2) = ((x^1)^2, 2x^1x^2)$

has Jacobean

$$Df(x^1, x^2) = \begin{pmatrix} 2x^1 & 2x^2 \\ 2x^2 & 2x^1 \end{pmatrix}$$

Whose rank is 2 on all of R^2 except the lines $x^2 = \pm x^1$. The rank is 1 on these lines except at $(0, 0)$ where it is zero.

If we compose f with diffeomorphisms, then the facts cited and the chain rule imply that the diffeomorphism have nonsingular Jacobeans. We say f has rank k on a set A, if it has rank for each $x \in A$.

Theorem [12] (Rank Theorem)

Let $A_0 \subset R^n$, $B_0 \subset R^m$ be open sets, $F : A_0 \rightarrow B_0$ be a $\mathrm{C^r}$ mapping , and suppose the rank of F on A_0 to be equal to k. If $a \in A_0$ and $b = F(a)$, then there exists open sets $A \subset A_0$ and $R \subset R_0$ with $a \in A$ and $b \in B$, and there exist $\mathrm{C^r}$diffeomorphism $G : A \rightarrow U$ (open) $\subset R^n$, $H : B \rightarrow V$(open)$\subset R^m$ such that H o Fo $\mathrm{G}^{-1}(\mathrm{U}) \subset V$ and such that this map has the simple form $HoFo\mathrm{G}^{-1}(x^1, \dots, x^n) = (x^1, \dots, x^k, 0, \dots, 0)$.

Differentiable manifolds

A differentiable manifold is a type of manifold that is locally similar enough to a linear space to allow one to do calculus. Any manifold can be describe by atlas (collection of charts).

In formal terms, a differentiable manifold is a topological manifold with a globally defined differential structure. Any topological manifold can be given a differential structure locally by using the homeomorphisms in its atlas and the standard differential structure on a linear space. To induce a global differential structure on the local coordinate system induced by the homeomorphisms, their composition on chart intersections in the atlas must be differentiable functions on the corresponding linear space. Differentiable manifold are very important in physics. Special kinds of differentiable manifolds form the basis for physical theories such as classical mechanics, general relativity.

Definition

A real -valued function f on R^n is differentiable (or infinitely differentiable, or smooth, or of class C^∞) provided all partial derivatives of f, of all orders, exist and are continuous.

Example

The function $f : X \rightarrow X^3, x \in R$ is C^∞ :

$$\frac{df}{dx} = 3x \ , \frac{d^2f}{dx^2} = 6x \ , \frac{d^3f}{dx^3} = 6 \ , \frac{d^4f}{dx^4} = 0 \ , \dots$$

Definition

Let M be a topological n-manifold. A coordinate chart (or just a chart) on M is a pair (U, φ), where U is an open subset of m and $\varphi: U \rightarrow \widetilde{U}$ is a homeomorphism from U to an open subset $\widetilde{U}$ = $\varphi(U) \subset R^n$. (Figure 1.10). By definition of a topological manifold, each point $p \in M$ is contained in the domain of some chart (U, φ). If $(p) = 0$, we say that the chart is centered at p · If (U, φ) is any chart whose domain contains, it's easy to obtain a new chart centered at p by subtracting the constant vector(p). Given a chart (U, φ), we call the set U a coordinate domain, or a coordinate neighborhood of each of its point. If in addition $\varphi(U)$ is an open ball in $\mathrm{R^n}$, then U is called a coordinate ball. The map φ is called a (local) coordinate map, and the component functions $(x^1, \dots, x^n)$ of φ, defined by$\varphi(p) = (x^1(p), \dots, x^n(p))$, are called local coordinates on U.

Definition (A coordinate chart)

A collection $A = \{(\varphi_i, U_i)\}_{i=I}$of coordinate charts with $M = \cup_i \mathrm{U_i}$, is called an atlas for M. Simply the set of charts is called an atlas.

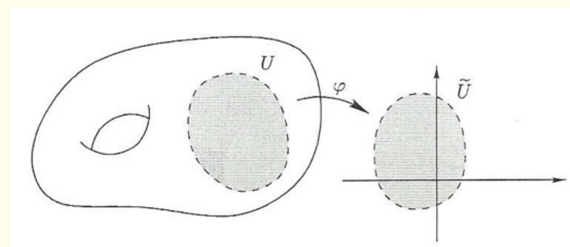

Figure 9

Definition

Given two overlapping charts, a translation cabe defined which goes from an open ball in $\mathrm{R^n}$ to the manifold and then back to another open ball in $\mathrm{R^n}$ the resultant map is called a change of coordinate, a coordinates transformation, a translation function, or a translation map. Simply the translation between choices of coordinates. (See figure 10).

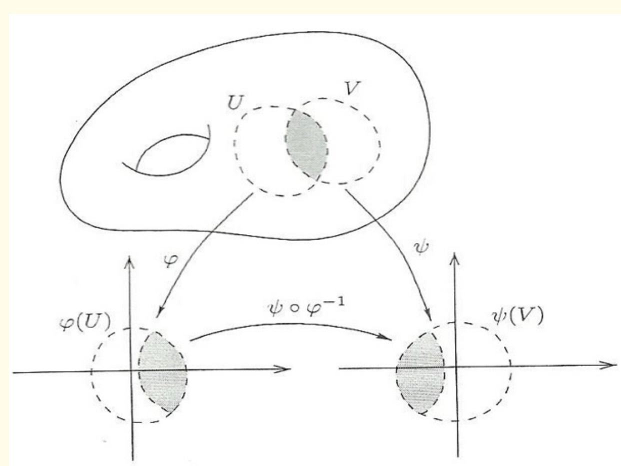

Figure 10

Definition

A differentiable or C^∞ (or smooth) structure on a topological manifold M is a family $\mathcal{U} = \{U_\alpha, \varphi_\alpha\}$ of coordinate neighborhoods such that :

- The U_α cover M.
- For any α, β the neighborhoods U_α, φ_α and V_β, φ_β are C^∞ compatible.
- Any coordinate neighborhood V, φ compatible with every $U_\alpha, \varphi_\alpha \in \mathcal{U}$ is itself in $\mathcal{U}$.

A C^∞ manifold is a topological manifold together with a C^∞- differential structure.

Theorem [7]

Let M be a Hausdorff space with a countable basis of open sets.

If $V = \{V_\beta, \psi_\beta\}$ is a covering of M by C^∞ - compatible coordinate neighborhood, then there is unique C^∞ structure on M containing these coordinate neighborhoods.

Example

(The Euclidean plan): The Euclidean plane E^2 becomes a metric space. Its Hausdorff and has countable basis of open sets; the choice of an origin and mutually perpendicular coordinate axes establish a homeomorphism (even isometric) $\psi : E^2 \to R^2$.

Thus we cover E^2 with a single coordinate neighborhood v, ψ with $V = E^2$, and $\psi(v) = R^2$. It follows not only that E^2 is a topological manifold, but by theorem (1.6.8) V, ψ determines a differentiable structure, so E^2 is a C^∞ manifold.

There are many other coordinate neighborhood on E^2 which are C^∞-compatible with V, ψ that is, which belong to the differentiable structure determined by V, ψ.

For example we may choose another rectangular Cartesian coordinate system $\acute{V}, \acute{\psi}$. Then it is shown in analytic geometry that the change of coordinates is given by linear, hence C^∞ (even analytic) function.

$$y^1 = x^1 cos\emptyset - x^2 sin\emptyset + h, y^2 = x^1 sin\emptyset + x^2 cos\emptyset + k.$$

Note that $V = \acute{V}$, but the coordinate neighborhood are not the same, since $\psi \neq \psi$, that is, the coordinates of each point are different for the two mapping.

Theorem [8]

Let M and N be C^∞ manifolds of dimension m and n. Then $M \times N$ is a C^∞ manifolds of dimension $m + n$ with C^∞ structure determined by coordinate neighborhood of the form $\{U \times V, \varphi \times \psi\}$, where U, φ and V, ψ are coordinate neighborhoods on M and N, respectively, and $\varphi \times \psi\ (p, q) = (\varphi_{(p)}, \psi(q))$ in $R^{m+n} = R^m \times R^n$.

Example

The tours $T^2 = s^1 \times s^1$, the product of two circle (See figure 13).

More generally, $T^n = s^1 \times, \ldots, s^n$, the n-fold product of circle is a C^∞ manifold obtained as a Cartesian product.

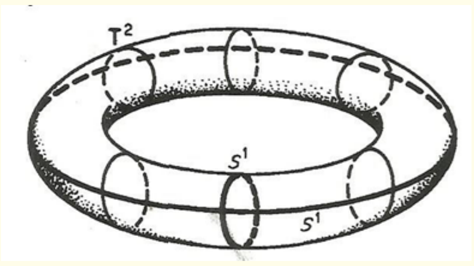

Figure 11

Differentiable function and mapping

Let f be a real-valued function defined on an open set W_f of a manifold M, in brief $f : W_f \to R$. If U, φ is a coordinate neighborhood on $\varphi(W_f \cap U)$ defined by $\hat{f} = f o \varphi^{-1}$, that is, so that $f(p) = \hat{f}\ (x^1(p), \ldots, x^n(p) = \hat{f}(\varphi(p))$ for all $p \in W_f \cap U$.

Definition

A C^∞ mapping $f: M \to N$ between C^∞ manifold is a diffeomorphism if it is a homeomorphism and f^{-1} is C^∞. M an N are diffeomorphic if there exist a diffeomorphism $f : M \to N$.

Rank of a mapping immersion

Let $f: N \to M$ be a differentiable mapping of φ manifold and let $p \in N$. If U, φ and V, ψ are coordinate neighborhoods of p and $f(p)$, respectively, and $f(U) \subset V$, Then we have a corresponding expression for f in local coordinate namely,

$$\hat{f} = \psi\ o\ f o \varphi^{-1} : \varphi(U) \to \psi(V).$$

Definition

The rank of f at p is defined to be the rank of $\hat{f}$ at $\varphi(p)$ of the Jacobean matrix

$$\begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n} \end{pmatrix}$$

Of the mapping

$\hat{f}(x^1, \ldots, x^n) = (f^1(x^1, \ldots, x^n), \ldots, f^m(x^1, \ldots, x^n))$ expression f in the local coordinates. This definition must be validate by showing that the rank is independent of the choice of the coordinates.

Example

$f : R \to R^2$ is given by $f(t) = \left(2\cos(t - \frac{1}{2}\pi), sin2(t - \frac{1}{2}\pi)\right)$

The image is a " figure eight " traversed in the sense shown (Figure 12 a) with the image point making a complete circuit starting at the origin as t goes from 0 at 2π.

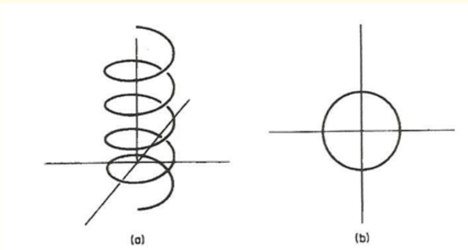

Figure 12

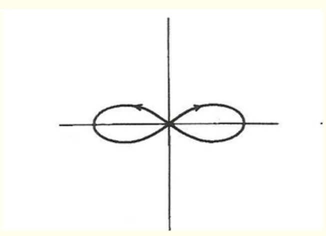

Figure 13

Sub manifolds

We begin the discussion of submanifold by introducing submanifold of R^n which form an important class of manifolds.

Definition

A subset $S \subset R^n$ is an m −dimensional submanifold of R^n of class C^k $(m \leq n)$,if, for each $p \in S$ there exist an open neighborhood V_p of p in R^n, an open set $\Omega_p \subset R^m$ and a homeomorphism $f_p: \Omega_p \to V_p \cap S$ which is C^k and regular in the sense that for every $a \in \Omega_p$, the differential $d_a f_p: R^m \to R^n$ is injective. The map $f_p: \Omega_p \to S$ is called a (local) parameterization of S around p .

Definition

A regular sub manifold of a C^∞ manifold $M$is any subspace N with the submanifold property and with the C^∞ structure that the corresponding preferred coordinate neighborhood determine on it.

Theorem [12]

If $F: N \to M$ is a one-to-one immersion and N is compact then F is an imbedding and $\tilde{N} = F(N)$ a regular submanifold.

Example

The map $F : R^n \to R$ defined by

$$f(x^1, \dots, x^n) = \sum_{i=1}^{n} (x^i)^2$$

Has rank 1 on $R^n - \{0\}$, which contains $F^{-1}(+1) = S^{n-1}$ is an $(n-1)$ −dimensional submanifold of R^n.

Definition (Riemannian Manifold)

A Riemannian manifold is a pair$(M.g)$ consisting of a smooth manifold M and a metric g on the tangent bundle, i.e., a smooth, symmetric positive definite $(0,2)$ -tensor field on M. The tensor g is called a Riemannian metric on M.

Two Riemannian manifolds $M_i, M_i)$ $(i = 1,2)$ are said to be isometric if there exists a diffeomorphism $\Phi : M_1 \to M_2$ such that $\Phi^* g_2 = g_1$.

Definition

Let (N, h) be a Riemannian manifold and M be a submanifold of N. Then the smooth tensor field $g: C_2^\infty(TM) \to C_0^\infty(TM)$ given by $g(X, Y): p \longmapsto h_b(X_p, Y_p)$.

Is a Riemannian metric on M called the induced metric on M in (N, h).

Example

The Euclidean metric $\langle , \rangle_{R^n}$ on R^n induces Riemannian metric on the following submanifold :

- The m-dimensional sphere $R^m \subset R^n$, with $n = m + 1$.
- The tangent bundle $TS^m \subset R^n$, where $n = 2(m + 1)$.
- The m-dimensional torus $T^m \subset R^n$, with $n = 2m$.

Theorem: (Fundental Theorem of Riemannian Geometry)

Let M be a Riemannian manifold, there exists a uniquely determined Riemannian connection on M.

Theorem [8]

A connected Riemannian manifold is a metric space with the metric $d(p, q) = infimum$ of the lengths of curves of class C^1 from P to q its metric space topology and manifold topology agree.

Lie group

Lie groups are important example of differentiable manifolds. The space R^n is$a$ C^∞ manifold and at the same time an Abelian group with group operation given by component wise addition. More over the algebraic and differentiable structures are related : $(x, y) \to x + y$ is a C^∞ mapping of the product manifold $R^n \times R^n$ onto R^n that is the group operation is differentiable. We also see that the mapping of R^n onto R^n given by taking each element x to its inverse – x is differentiable.

Now let G be a group which is at the same time a differentiable manifold. For $x, y \in G$ let xy denote their product and x^{-1} the inverse of x.

Definition

G is a lie group provided the mapping of $G \times G \rightarrow$ defined by $(x, y) \rightarrow xy$ and the mapping of $G \rightarrow G$ defined by $x \rightarrow x^{-1}$ are both C^{∞} mapping.

Lemma [9]

Suppose G is a smooth manifold with a group structure such that the map $G \times G \rightarrow G$ given by $(g, h) \rightarrow gh^{-1}$ is smooth then G is a lie group.

Definition

A group is a set G with a binary operation denoted here by concatenation such that

- $(x, y)z = x\,(yz)\, for\ \ all\ \ x, y, z \in G$
- There is an identity $e \in G$ satisfying $e\,x = x = x$e for all $x \in G$
- Each $x \in G$ has inverse x^{-1} satisfying $x\,x^{-1} = x^{-1}\,x = e$

Examples

- The real number field $\mathbb{R}$ and Euclidean space $\mathbb{R}^n$ are Lie group under addition, because the coordinates of x, y are smooth (linear) functions of (x, y).
- The set $\mathbb{R}^*$ of nonzero real numbers is a 1-dimentional lie group under multiplication (in fact it is exactly $Gl(1, \mathbb{R})$, if we identify a 1×1 matrix with the corresponding real number. The subset $\mathbb{R}^+$ of positive real numbers is an open subgroup, and is thus itself a 1-dimensional Lie group.
- The set $\mathbb{C}^*$ of nonzero complex numbers is a 2-dimensional Lie group under complex multiplication which can be identified with $Gl(1, \mathbb{C})$
- If V is any real or complex vector space we let $Gl(V)$ denote the group of invariable linear transformation from V to itself. If V is finite dimensional, any basis for V determines an isomorphism of $Gl(V)$ with Gl(n,$\mathbb{R}$) or $Gl(n, \mathbb{C})$ with $n = dimV$ so $Gl(V)$ is a Lie group.

Lemma [12]

Let $F: A \rightarrow M$ be a C^{∞} mapping of C^{∞} manifold and suppose $F(A) \subset N$, N being a regular submanifold of M. Then F is C^{∞} as a mapping into N.

Definition

Let $F : G_1 \rightarrow G_2$ be an algebraic homeomorphism of Lie groups G_1 and G_2. We shall call F a homeomorphism (of Lie groups) if F is also a C^{∞} mapping.

Definition

A (Lie) subgroup H of a Lie group G will mean any algebraic subgroup which is a sub manifold and is a Lie group with its C^{∞} structure as an (immersed) submanifold.

The action of a lie group on a manifold

Definition

Let G be a group and X a set. Then G is said to act on X (on the left) if there is a mapping $\theta : G \times X \rightarrow X$ satisfying two condition:

If e is the identity element of then $\theta(e, x) = x\ for\ all\ x \in X$.

If $g_1, g_2 \in G$, then

$\theta\big(g_1, \theta(g_2, x)\big) = \theta(g_1 g_2, x)\ for\ all\ x \in X$.

When G is a topological group, X is a topological space, and θ is continuous, then the action is called continuous. When G is a lie group, X is a C^{∞} manifold and θ is a C^{∞} we speak of a C^{∞} action.

Theorem [12]

The natural map $\pi : G \rightarrow G/H$ taking each element of G to its orbit, that is to its left coset is not only continuous, but open. G/H is hausdorff if and only if H is closed.

Main Results

- Concept of a manifold is central to many parts of geometry and modern mathematical physics because it allows complicated structures to be described in terms of well-understood topological properties of simpler spaces. Manifolds naturally arise as solution sets of systems of equations and as graphs of functions.
- A manifold is the multidimensional analog of a surface. All the smooth surfaces (i.e., no hard edges or points) that you are familiar with are Riemannian manifolds of dimension 2. That means that measurements on the surface are determined by how that surface sits in space.
- The primary motivation of manifolds is that they be locally Euclidean. Every point in the manifold looks Euclidean when zoomed in. It is approximated by a Euclidean tangent space.
- Euclidean spaces are very nice. They are Hilbert spaces (complete inner product vector spaces). This means that there are notions of distance between points, angles between lines, limits of sequences of points, and that these properties don't change under translations and rotations. As metric spaces they are second countable; the topology induced has a countable basis.

Conclusion

There are very special Riemannian manifolds, in fact locally symmetric spaces can be defined as Riemannian manifolds such that the covariant derivative of the curvature is zero.

Many interesting moduli space in algebraic geometry and number theory are given by locally symmetric spaces.

They can be systematically constructed using Lie groups, algebraic groups and arithmetic group and can be used to study such groups. Geometry of locally symmetric spaces are closely related to algebraic structures in Lie groups and algebraic groups for locally symmetric spaces of finite volume the reduction theory is crucial to problems both in geometry and analysis on locally symmetric spaces.

They are natural spaces for Lie groups and arithmetic groups to act on, and hence give rise to natural representations of the lie groups.

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