Acta Scientific COMPUTER SCIENCES

Volume 5 Issue 12 December 2023

Research Article

Implicit Three-Step Multi-Derivative Algorithm for the Solution of Second Order Ordinary Differential Equations

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Received: November 23, 2023

Published: November 30, 2023

Abstract

Our aim is to construct an implicit three step method with multiderivative to handle general second order initial value problems of ordinary differential equations (ODEs) directly. The study provides the use of both collocation and interpolation techniques to obtain the method. In deriving the method, power series and Bernstein polynomial in combine was used as an approximate solution. An order six, consistent, zero-stable method and hence convergent is obtained. The main predictor was developed using the same approach and is of equal order as the corrector. Absolute error of the method obtained with some test problems showed an improve accuracy over the existing methods in the reviewed literature.

Keywords: Second Order; Ordinary Differential Equation; Multi-derivative; Interpolation; Collocation; Predictor-Corrector; Consistency; Zero Stable

Introduction

In this paper, we shall consider the solution of general second order differential equation of the form.

$$y'' = f(x, y, y') \quad y(b) = y_0; y'(b) = y_1, \quad f \in c^2(a\ ,\ b) \tag{1}$$

Many literatures have shown that (1) is conventinally reduced to system of first order ordinary differential equations in attempts to solve them and methods of first order ordinary differential equations are used to solve them. Due to the dimension of the problem after it has been reduced to a system of first order ordinary differential equations (ODEs), also the reduced systems of ODEs are not well posed unlike the given problem. The approach wastes a lot of computer time and human efforts, hence there is a great need to develop new and efficient algorithms to handle problem (1) directly without any reduction to its equivalent system of first order ODEs.

Several authors have also solved problem (1) through predictor corrector mode (PC) of implementations; among them are [7,22]. Although the implementation of the methods in a PC mode yields good accuracy, the approach is more costly to implement, for instance PC routines are very complicated to write, since they require special techniques for supplying starting values and also predicting all the off grid points present in the method which leads to longer computer time and human efforts to handle their approach. The accuracy of these methods in terms of error is not encouraging as thus can be improved by developing a non-hybrid method having same of the predictor with that of the Corrector which is our focus in this work.

In developing these methods capable of solving (1), various kinds of basis functions such as the Hermite polynomial, chebyshev polynomial, Legendre polynomials, the monomals, Bernstein polynomial, power series to mention but a few have been employed in literatures for the development of linear multistep method to solve (1). Its was observed that in the development of most methods, power series was used as the basis function. [3,5,8,10,12,18,20], to mention but a few all consider a power series approximate solution for developing there various methods. Few among scholars who consider Chebyshev as the basis function are [6,11,13-15] while [1,2,9,10] are among the scholars who employ Legendre polynomial of first kind as there basis function for there methods. Its was also notices that polynomials of different kinds can be combined as a basis function for the development of methods capable of solving (1). Scholars like [16], proposed approximate solution which combined power series polynomial and exponential function as a basis function for the solution of first other ordinary differential equations (stiff equations), [7] proposed an approximate solution which combined Chebyshev and Legendre polynomials for solving general second order odrdinary differential equation. In this work, a combination of power series and Bernstein polynomial was used as basis function in generating the interpolation and collocation equations for the development of the method for the solution of (1) with higher derivative.

Development of the method

Considering an approximation of the form;

$$y(x) = \sum_{j=0}^{k+2} a_j x^j + \sum_{j=k+2}^{w} a_j B_{j,w}(x) \tag{2}$$

In order to obtain (6) is adopted, where is the number of collocation, is the number of interpolation points carefully chosen and $w = 2u + v - 1$ with $k = 3$ was considered in this work. a_j's is the coefficient to be determined and $B_{(j,w)}(x)$ is the Bernstein polynomial to be obtained from the equation below.

$$B_{j,w}(x) = \binom{w}{j} x^j (1-x)^{w-j}, j = k+3, \ldots, w \tag{3}$$

Where the binomial coefficient is

$$\binom{w}{j} = \frac{w!}{j(w-j)!}$$

Equation (4) and (5) are the second and third derivative of (2) given below;

$$y'' = \sum_{j=2}^{k+2} j(j-1)a_j x^{j-2} + \sum_{j=k+2}^{w} a_j B''_{j,w}(x) = f(x, y, y') \tag{4}$$

$$y'''^{(x)} = \sum_{j=3}^{k+2} j(j-1)(j-2)a_j x^{j-3} + \sum_{j=k+2}^{w} a_j B_{j,w}(x) = f(x, y, y', y'') \tag{5}$$

Equations (2) was interpolated while (4) and (5) were collocated to acquire the needed method for step number k. Collocated (4) and (5) at $x_{(n+j,)}$ j=0,1,3and interpolated (2) at $x_{(n+j,)}$ j=1,2 gives a system of equations which is solved using Gaussian elimination method to get the values of the unknown parameters a'_j s,j=0(1)7.. The is obtained are then substituted into (2) to obtain the continuous form of the method;

$$y_{n+k}(x) = \sum_{j=0}^{k-1} \alpha_j y_{n+j}(x) + h^2 \sum_{j=0}^{k} \beta_j f_{n+j}(x) + h^3 \sum_{j=0}^{k} \omega_j q_{n+j}(x) \tag{6}$$

Where α_j, β_j and ω_j are the continuous coeffcients.

$$y_{n+j} \simeq y(x_{n+j});\ f_{n+j} = f\left(x_{n+j}, y_{n+j}, y'_{n+j}\right);\ q_{n+j} = q\left(x_{n+j}, y_{n+j}, y'_{n+j}, y''_{n+j}\right);\ h$$

his the stepsize and is the step number of the method. The first derivative of equation (6) is given below.

$$y'_{n+k}(x) = \frac{1}{h} \sum_{j=k-2}^{k-1} \alpha'_j y_{n+j}(x) + h \sum_{j=0}^{k} \beta'_j f_{n+j}(x) + h^2 \sum_{j=0}^{k} \omega'_j q_{n+j}(x) \tag{7}$$

Using the transformation in [22];

$$t = \frac{x - x_{n+k-1}}{h}$$

$$\frac{dt}{dx} = \frac{1}{h}$$

y_{n+j}, $f_{n+j}$and q_{n+j} are the coeffcients obtained as.

$$\alpha_2(t) = t \quad \text{-----}(8)$$

$$\alpha_3(t) = -(t+1) \quad \text{-----}(9)$$

$$\beta_0(t) = h^2\left(\frac{4}{567}t^7 + \frac{19}{810}t^6 - \frac{4}{135}t^5 - \frac{19}{162}t^4 + \frac{4}{81}t^3 + \frac{19}{54}t^2 + \frac{437}{1890}t\right) \quad \text{-------}(10)$$

$$\beta_1(t) = h^2\left(-\frac{1}{168}t^7 - \frac{1}{60}t^6 + \frac{3}{80}t^5 + \frac{1}{12}t^4 - \frac{1}{6}t^3 + \frac{113}{560}t\right) \text{-------}(11)$$

$$\beta_3(t) = h^2\left(-\frac{5}{4536}t^7 - \frac{11}{1620}t^6 - \frac{17}{2160}t^5 + \frac{11}{324}t^4 + \frac{19}{162}t^3 + \frac{4}{27}t^2 + \frac{1013}{15120}t\right)$$

$$\text{--------------------}\ (12)$$

$$\omega_0(t) = h^3\left(-\frac{1}{378}t^7 + \frac{1}{135}t^6 - \frac{1}{90}t^5 - \frac{1}{27}t^4 + \frac{1}{54}t^3 + \frac{1}{9}t^2 + \frac{1}{14}t\right) \text{-------}\ (13)$$

$$\omega_1(t) = h^3\left(\frac{1}{168}t^7 + \frac{1}{40}t^6 - \frac{1}{80}t^5 - \frac{7}{48}t^4 + \frac{1}{2}t^2 + \frac{27}{70}t\right) \text{-----}(14)$$

$$\omega_3(t) = h^3\left(\frac{1}{1512}t^7 + \frac{1}{216}t^6 + \frac{7}{720}t^5 - \frac{1}{432}t^4 - \frac{1}{27}t^3 - \frac{1}{18}t^2 - \frac{67}{2520}t\right) \text{---------}(15)$$

Substitute into (6) and Putting t = 1 in (8-15) produced our method with its first derivative.

$$y_{n+3} = 2y_{n+2} - y_{n+1} + \frac{209}{405}h^2 f_n + \frac{2}{15}h^2 f_{n+1} + \frac{142}{405}h^2 f_{n+3} + \frac{22}{135}h^3 q_n$$
$$+ \frac{91}{120}h^3 q_{n+1} - \frac{23}{216}h^3 q_{n+3} - -(16)$$

$$y'_{n+3} = \frac{1}{45360h}\begin{pmatrix} 9456h^3 q_n + 42255h^3 q_{n+1} - 8031h^3 q_{n+3} + 29752h^2 f_n + 3672 f_{n+1} \\ +34616 f_{n+3} - 45360 y_{n+1} + 45360 y_{n+2} \end{pmatrix}$$
$$- - - - (17)$$

Predictor for the Method

For the main predictor, equation (2), (4) and (5) were interpolated and collocated at grid points and respectively. The main predictor and it derivative for the method were derived using the same approximate solution (2). The procedure to getting the predictor is the same with the main method with same points of interpolation and different points of collocations. Below is the developed predictor and its first derivative

$$y_{n+3} = 2y_{n+2} - y_{n+1} + \frac{61}{120}h^2 f_n + \frac{11}{15}h^2 f_{n+1} - \frac{29}{120}h^2 f_{n+2}$$
$$+ \frac{3}{20}h^2 q_n + h^3 d_{n+1} + \frac{3}{5}h^3 d_{n+2} - -(18)$$

$$y'_{n+3} = \frac{1}{1680h}\begin{pmatrix} 1222h^3 d_n + 7688h^3 d_{n+1} + 4400 d_{n+2} + \\ 4107 f_n + 4536h^2 f_{n+1} - 6123h^2 f_{n+2} - 1680 y_{n+1} + 1680 y_{n+2} \end{pmatrix}$$
$$- - -(19)$$

Analysis of the method

Order and Error constant

We adopt the method proposed by [4,19] to obtain the order of the method (16), (17), (18), (19) as $(6,6,6,6)^T$ and error constant as

$$\left(\frac{1261}{302400}, \frac{29483}{282240}, \frac{659}{302400}, \frac{2273}{201600}\right)$$

Consistency

Definition: The method is said to be consistent if it has order of at least one. If we define the first and second characteristic polynomial

$$\rho(x) = \sum_{j=0}^{k} \alpha_j z^j \ \text{-------------}\ (20)$$

$$\sigma(x) = \sum_{j=0}^{k} \alpha_j z^j$$
$$\text{-------------}\ (21)$$

Where z is the principal root, $\alpha_j \neq 0$ *and* $\alpha_0^2 + \beta_0^2 \neq 0$

Definition: The linear multistep method (6) is said to be consistent, if it satisfies the following conditions.

- The order $\rho \geq 1$
- $\sum_{j=0}^{k} \alpha_j = 0$
- $\rho(1) = \rho'(1) = 0$
- $\rho''(1) = 2\ !\sigma(1)$

For our method

Condition (i) is satisfied since the scheme is of order 6.

Condition (ii) is satisfied since

$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0;\ 0 + 1 - 2 + 1 = 0.$

Condition (iii) is satisfied since

$\rho(r) = r^3 - 2r^2 + r \text{ and } \rho'(r) = 3r^2 - 4r + 1.$

Where r =1; $\rho(r) = \rho'(r) = 0$

Condition (iv) is satisfied since

$\rho''(r) = 6r - 4 \text{ and } \sigma(r) = \frac{1}{405}(209 + 54r + 142r^3)$

Where r =1;

$\sigma(1) = 2! \times \left(\frac{209}{405} + \frac{2}{15} + \frac{142}{405}\right) = 2 \times \frac{405}{405} = 2 \times 1 = 2$

Therefore $\rho''(r) = 2!\sigma(r) = 2$

The method is consistent, since the four condition are satisfied.

Zero stability

Definition: According to [19], linear multistep method is said to be zero-stable, if no root of the first characteristics polynomial $\rho(r)$ has modulus greater than one and if every root of modulus one has multiplicity not greater than one. The method is zero stable when no root of the first characteristics polynomial has modulus greater than one that is $|r| \leq 1$.

A method is zero stable if $\rho(x) = \sum_{j=0}^{k} \alpha_j = 0,$, where α_j are the coefficients of $\sum_{j=0}^{k} \alpha_j y_{n+j}$

$$\sum_{j=0}^{k} \alpha_j = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 0 + 1 - 2 + 1 = 0$$

Given the first characteristic polynomial of (16) as: $\rho(r) = r^3 - 2r^2 + r^1 = 0$. Solving $\rho(r), r = 0,0,1.$ Hence our method is zero stable.

Convergence

Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent [4].

Our method is convergent since it satisfies both the consistency and zero stability conditions.

Numerical problems

In this section, the implementation of the method in solving initial value problems (IVPs) of second order ordinary differential equation is shown. The new developed method is tested on some problems to determine the performance of the newly developed method.

Problem 1

$y''(x) = x(y')^2, y(0) = 1, y'(0) = \frac{1}{2}, h = \frac{1}{320}$

Exact solution: $y(x) = 1 + \frac{1}{x}\ln\left(\frac{2+x}{2-x}\right)$

Problem 2

$y''(x) = y', y(0) = 0, y'(1) = -1, h = \frac{1}{10}$

Exact solution: $y(x) = 1 - e^{-x}$

Problem 3

$y''(x) = 2y^3, y(1) = 1, y'(1) = -1, h = \frac{1}{10}$

Exact solution: $y(x) = \frac{1}{x}$

Problem 4

Resonance vibration of a machine

A spring with a mass of 2 kg has natural length m. A force of 5N is required to maintain it stretched to a length of m. If the spring is stretched to a length of m and then released with initial velocity 0, and the position of the mass at any time.

$k(0.2) = 25.6$

So $k = \frac{25.6}{0.2}, k = 128$

Using this value of the spring constant k, together with m = 2 then, we have

$2\frac{d^2x}{dx^2} + 128x = 0$

$x(t) = C_1 cos8t + C_2 cos8t$

$x'(t) = -8C_1 Sin8t + 8C_2 cos8t$

Since the initial velocity is given as as x' (0)=0, we have C_2= 0 and so the solution is

$x(t) = \frac{1}{5}\cos(8t)$

Problem 5

Resonance vibration of a machine

A stamping machine applies hammering forces on metal sheets by a die attached to the plunger moves vertically up and down by a fly wheel makes the impact force on the sheet metal and therefore the supporting base, intermitted and cyclic. The bearing base on which the metal sheet is situated has a mass =2000 kg. The force acting on the base follow a function:

f(t) =2000sin(10t) in which t= time in seconds.

The base is supported by an elastic pad with an equivalent spring constant

$k = \quad 2 * \quad 10^5\ N/M$

Determine the differential equation for the instantaneous position of the base y(t) if the base is initially depressed down by an amount 0.005m solution.

Solution: The mass spring system above is modeled as differential equation as:

The Bearing base mass = 2000kg

Spring constant $k = 2 * 10^5 N/M$.

Force (ma) in the metal sheet = $m\frac{d^2y}{dt^2} = my''$

i.e ma = my" = 2000 Sin (10t); where a = y".

Initial condition on the system are

$y(t_0) = y_0; y'(t_0) = y'(0); t_0 = 0, y_0' = 0.005$

Therefore, the governing equation for the instantaneous position of the base y(t) is giving by

$$my'' + ky = f(t);\ y(t_0) = y_0, y'(t_0) = y_0'$$

Theoretical Solution:

$$y(t) = \frac{1}{200}cos(1\,0t) + \frac{1}{200}sin(1\,0t) - \frac{t}{20}cos(1\,0t)$$

	Exact Result	Computed Result	Error in our method	Error in [1], p = 8
0.100	1.050041729278491268	1.050041729278491134	1.33524e-16	1.957046e-13
0.200	1.100335347731075581	1.100335347731075370	2.10015e-16	6.039897e-13
0.300	1.151140435936466805	1.151140435936465940	8.65002e-16	1.261598e-12
0.400	1.405465108108164382	1.202732554054081080	1.110223e-15	3.715303e-12
0.500	1.255412811882995342	1.255412811882993787	1.554312e-15	7.918892e-12
0.600	1.309519604203111715	1.309519604203087290	2.442491e-14	1.416178e-11
0.700	1.365443754271396169	1.365443754271365082	3.108624e-14	3.616015e-11
0.800	1.423648930193601807	1.423648930193568500	3.330669e-14	7.472525e-11
0.900	1.484700278594051742	1.484700278593629856	4.218847e-13	1.335141e-10
1.000	1.549306144334054846	1.549306144328725774	5.329071e-12	4.316861e-10

Table 1: Comparison of our results with that of [1] for Problem 1.

	Error in our method	Error in [22]	Error in [21]
0.003125	2.220446e-16	-	-
0.006250	0.000000e+00	9.325873e-15	5.2e-15
0.009375	2.220446e-16	1.865175e-14	5.0e-15
0.012500	2.220446e-16	2.797762e-14	9.9e-15
0.015625	4.440892e-16	3.730349e-14	1.6e-14

Table 2: Comparison of our results with that of [21,22] for Problem 1.

	Exact Result	Computed Result	Error in our method	Error in [17]
0.100	-0.10517091807564762	-0.10517091807564890	1.281961e-15	7.609423e-08
0.200	-0.22140275816016983	-0.22140275816025894	8.911274e-14	1.674320e-07
0.300	-0.34985880757600310	-0.34985880757618721	1.841097e-13	2.603737e-07
0.400	-0.49182469764127031	-0.49182470273003031	5.088760e-09	3.719340e-07
0.500	-0.64872127070012814	-0.64872131151049814	4.081037e-08	4.854533e-07
0.600	-0.82211880039050897	-0.82211888239086897	8.200036e-08	6.217134e-07
0.700	-1.01375270747047652	-1.01375284315647652	1.356860e-07	7.603662e-07
0.800	-1.22554092849246760	-1.22554118049446760	2.520020e-07	9.267947e-07
0.900	-1.45960311115694966	-1.45960349101744966	3.798605e-07	1.096145e-07
1.000	-1.71828182845904523	-1.71828235675004523	5.282910e-07	1.299421e-06

Table 3: Comparison of our results with that of [17] for Problem 2.

	Exact Result	Computed Result	Error in our method	Error in [9]
1.1	0.909090909090909090	0.909090909056353090	3.45560e-11	1.660347e-10
1.2	0.833333333333333333	0.833333333258887133	7.44462e-11	8.239573e-10
1.3	0.769230769230769231	0.769230768485099230	7.45670e-10	6.832158e-07
1.4	0.714285714285714286	0.714285713640034285	6.45680e-10	1.389440e-06
1.5	0.666666666666666667	0.666666666132344666	5.34322e-10	2.280078e-06
1.6	0.625000000000000000	0.624999999497750000	5.02250e-10	
1.7	0.588235294117647058	0.588235289661147058	4.45650e-09	
1.8	0.555555555555555555	0.555555552098955555	3.45660e-09	
1.9	0.526315789473684210	0.526315786228944210	3.24474e-09	
2.0	0.500000000000000000	0.499999997543400000	2.45660e-09	

Table 4: Comparison of our results with that of [9] for Problem 3.

	Exact Result	Computed Result	Error in our method
0.01	0.199360341260523880	0.199360341333333330	7.280945e-11
0.02	0.197445456675125390	0.197445461333333350	4.658208e-09
0.03	0.194267594970405940	0.194267604163697790	9.193292e-09
0.04	0.189847083616488170	0.189843051582907320	4.032034e-06
0.05	0.184212198800577020	0.184179965002135900	3.223380e-05
0.06	0.177398984555856840	0.177334814225734660	6.417033e-05
0.07	0.169451022202683210	0.169346422678079560	1.045995e-04
0.08	0.160419151576858540	0.160227866986046490	1.912846e-04
0.09	0.150361145828179000	0.150076709263508990	2.844366e-04
0.10	0.139341341869433090	0.138952299166472810	3.890427e-04

Table 5: Exact results and computed results of our method for Problem 4.

	Exact Result	Computed Result	Error in our method
0.01	0.004976685826985256	0.004976683325000000	2.501985e-09
0.02	0.004913612965340273	0.004913533066666666	7.989867e-08
0.03	0.004821278745246320	0.004821113024502985	1.657207e-07
0.04	0.004710274693551908	0.004709928073643812	3.466199e-07
0.05	0.004591084097746946	0.004590250429872411	8.336679e-07
0.06	0.004473883596794534	0.004472884057915152	9.995389e-07
0.07	0.004473883596794534	0.004472884057915152	1.017588e-06
0.08	0.004283487163844781	0.004283686024564759	1.988607e-07
0.09	0.004227439532272750	0.004230316628447344	2.877096e-06
0.10	0.004207354924039482	0.004213679652903020	6.324729e-06

Table 6: Exact results and computed results of our method for Problem 5.

Discussion of Results

An implicit three step multi-derivative algorithm for solving second order ordinary differential equation was studied in this research.

In Tables 1, 2, 3 and 4 above, our new method produced better accuracy compared to that of [1] (see Table 1). Our method also outmatches the method proposed by [21,22] (see Table 2) in terms of accuracy. The new method also outperform that of [17] (see Table 3). Our new method produced better accuracy compared to that of [9] (see Table 4).

Table 5 and 6 above, displays the result of problem 4 and 5 which are real life problem of second order. The computed compared favorably well with the exact.

Conclusions

The derivation of a new method with multi-derivative through interpolation and collocation approach for solving second order initial value problems of ODEs has been examined in this paper. The analysis of the basic properties shows that the method is consistent, zero stable and hence convergent. The new method performed favourably with the existing methods compared with in terms of accuracy. These are evidently shown in Table 1-4. Hence, it can be used to solve all kinds of second order initial value problems directly without necessarily resolving such an equation to a system of first order ordinary differential equations. In future work, the method can be modify to handle boundary value problems. However, further research could extend the method to direct solution of higher order general ODEs.

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