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Review Article

Root Systems, Cartan Matrix, Dynkin Diagrams in Classification of Symmetric Spaces

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Abstract

In this study we have introduced Riemannian Manifold, Lie groups, Lie algebras, and root systems which help to give further understanding of symmetric spaces and some of their algebraic and topological properties which help in classification and many applications of symmetric spaces. I address and explore the basic concept of a root system. First, its origins in the theory of Lie algebras are introduced and then an axiomatic definition is provided. Bases, Weyl groups, and the transitive action of the latter on the former are explained Finally, the Cartan matrix and Dynkin diagram are introduced to suggest the multiple applications of root systems to other fields of study and their classification.

Keywords: Lie Groups; Lie Algebras; Topological Spaces; Metric Spaces; Topological Manifold; Riemannian Manifold; Root Systems; Symmetric Spaces; Weyl group; CatanMatrix and Dynkin Diagrams

Introduction

Nevertheless when introducing symmetric spaces one cannot ignore the fact that a symmetric space can be introduced as a homogeneous space where is it's group of isometries, which is a Lie group, and is the isotropy subgroup. Many properties of symmetric spaces can be studied through their Lie algebras and root systems, and specially the problem of classification of symmetric spaces. Cartan solved most of this problem where he gave a full classification of symmetric spaces in terms of their Lie algebras which are related to root systems. Since then many authors contributed their efforts to developing this issue, and introduced Various applications of Lie algebras and symmetric spaces in different fields, especially in physics. For example Dyson [1] recognized that the integration manifolds in random matrix theory are symmetric spaces, it was found that the classification of new matrix symmetry classes in terms of Cartan's symmetric spaces corresponds to ten of the eleven classes of symmetric spaces in Cartan's classification. Also Olshanetsky and Perelomov [2] demonstrated the deep connection between some quantum integrable systems and root systems of Lie algebras which led to symmetric spaces, where this work shows that the symmetry of underlying root systems makes Calogero – Sutherland models for some values of the coupling constants integrals. Also Olshanetsky and Perelomov showed that the dynamics of quantum enterable systems is related to free diffusion on symmetric spaces.

To anchor our discussion of root systems, let us begin with a general overview of their occurrence in the theory of Lie algebras. A Lie algebramaybeunderstoodasavectorspacewithanadditionalbilinearoperationknownasthe commutator [,] defined for all elements and satisfying certain properties. Alien algebra is called simple fits only ideals are itself and0.Andspecifically the derived 'algebra. (This is analogous to the commutator subgroup of a group being nontrivial). Let the Lie algebra L be semi simple, i.e. decomposable as the direct product of simple Lie algebras. Then we define a total sub algebras the span of some semi simple elements of L. It is natural to consider a maximal toral sub algebrawhich is not properly contained in any other. It turns out that may then written as the direct sum of and the subspaceswhere ranges over all element of. The nonzero for which are called the roots of L relative to Root systems thus provide a relatively uncomplicated way of completely characterizing simple and semi simple Lie algebras. It is the goal of this paper to show that root systems may be themselves completely characterized by their Cartan matrices.

Lie groups

Definition 2.1

A Lie group G is a group satisfying the well-known axioms of group, besides the mappings $G \times G \rightarrow G$ and $C \rightarrow G^{-1}$ defined by $(x,y) \rightarrow xy$ and $x \rightarrow x^{-1}$.

Respectively are both C^{∞}. This definition implies that the Lie group is a differentiable manifold. Lie groups are very important due to the fact that, their algebraic properties derive from group axioms, and their geometric properties derive from the identification of group operations with points in a topological space.

Examples 2.2

- The set C* of nonzero complex numbers is a 2-dimensional Lie group under complex multiplication which can be identified with Gl(1,C).
- The set GL (n,R) of nonsingular n × n matrices is a group with respect to matrix multiplication.. An n × n matrix X is nonsingular if and only X ≠ 0. if det X,Y∈ GL (n,R) If then both the maps (X,Y) →XY and x→ x^{-1} are C^{∞}. Thus GL (n,R) is a Lie group.
- The Euclidean space under addition is a group, endowed with the smooth operations (x,y)→x+y and x→x^{-1} ∀ x,y ∈R^n forms a Lie group.[2]

There are many other examples for Lie groups and their applications which can be seen in various references like [3-6]. The matrices in can be represented as

$$M = \exp\left(\sum_i t^i X_i\right) \quad (2.1)$$

Where X_l are the generators of what is called the Lie algebra of the Lie group and t^l are real parameters. For a Lie group the tangent space at the origin is spanned by the generators, considered as vector fields which are expressed as $X = X^i(x)\frac{\partial}{\partial x^I}$, where the partial derivatives $\frac{\partial}{\partial x^I}$ form a basis for the vector field. If X is a generator of a lie group then X onto $\exp^{tX}$ is the exponential map, which is a one - parameter subgroup, defining a curve c(t) in the group manifold. For the curve c(t) the tangent vector at the origin is given by The matrix exponential is very useful because it is always nonsingular since $\frac{d}{dt}e^{tX}|\, t = X$ (2.2).

Lie algebra

A Lie algebra is an algebraic structure whose main use is in studying geometric objects such as Lie groups and differentiable manifolds.

Definition 3.1

A Lie algebra is a pair (V, [,]) where Vis a vector space, and [,] is a Lie bracket, [,]: V × V → V satisfying :

- [v, w] = −[w, v] skew-symmetric.
- [av + bu, w] = a[v, w] + b[u, w]a bilinear.
- [v, [w, u]] + [w, [u, v]] + [u, [v, w]] = 0, For all, uandw ∈ V. (Bianchi identity).

A Lie Bracket is a binary operation [,] on a vector space.

Example 3.2

The Lie algebra**g** of R^n as a Lie group, is again R^n, where [X, Y] = 0∀ X, Y ∈ G Thus the Lie bracket for the Lie algebra of any abelian group is zero.

Example 3.3

Let V = R^3, [,]: $R^3 \times R^3 \rightarrow R^3$as proved that it is a Lie algebra.

Example 3.4

Let Ω (M) be the set of all vector fields on a manifold. Define
[v, w] = vw − wv,
Then [v, w] is a Lie bracket.

A homeomorphism of Lie algebra l is a linear map, $\varphi : \ell \rightarrow \acute{\ell}$, preserving the bracket. This means that $\varphi[\ell 1, \ell 2] = [\varphi(\ell_1), \varphi\, \ell]$ for any $(\ell_1, \ell_2) \in \ell \times \ell$. A Lie subalgebra of Lie algebra l is a subvector space η such that [η, η] ⊆ η. An ideal of l is a Lie subalgebra η such that [η, ℓ] ⊆ η.

Definition 3.5

A vector subspace η of a Lie algebra l is called a Lie subalgebra if [η, ℓ] ⊆ η.

Theorem[9]3.6

Let G be a Lie group and l its Lie algebra

- If H is a Lie subgroup of G, η is a Lie subalgebra of l.
- If η is a Lie subalgebra, there exists a unique Lie subgroup H of G such that the Lie algebra of H is isomorphic to η.

Properties of a Lie algebra 3.7

We now turn to the properties of a Lie algebra. These are derived from the properties of a Lie group. A Lie algebra has three properties:

- The operators in a Lie algebra form a linear vector space
- The operators closed under commutation: the commutator of two operators is in the Lie algebra;
- The operators satisfy the Jacobi identity.

Topological spaces

Topological spaces are mathematical structures that allow the formal definition of concepts such as convergence, connectedness and continuity. They appear virtually in every branch of modern mathematic and are central unifying notions. The branch of mathematics that studies topological spaces in their own right is called topology.

Definition of topological spaces 4.1

Let X be a set, let T be a collection of subsets such that

- The union of a family of sets which are elements of T belongs to T.
- The intersection of a finite family of sets which are elements of T belongs to T.
- The empty set ∅ and the whole X belong to T. Then T is called a topological structure or just a topology in X. The pair (X,T) is called a topological space.

The element of X is called point of this topological space.

The element of T is called open set of the topological space (X,T). The Conditions in the definition above are called the axioms of topological structure.

Examples 4.2

- A discrete topological space is a set with the topological structure which consists of all the subsets.
- The Euclidean spaces R^n can be given a topology in the usual topology on R^n, the basic open sets are the open balls.

Metric spaces Definition 5.1

A metric space is a set with a function that satisfies.

d: X × X → R+ that satisfies.

- $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$
- $d(x, y) = d(y, x) = 0$ for every $x, y \in X$
- $d(x, y) + d(y,z) \geq d(x,z)$ triangle inequality The pair (X, d) is called metric space.

Example 5.2

The usual metric on C (complex numbers) is the Euclidean metric determined by the modulus function $((z, w)) \to$

$|z - w|$. It is of course an extension to G × G of the Euclidean metric on R. We shall assume that G is endowed with it unless we state otherwise.

Theorem 5.3 [8]

Suppose(X, d) is a metric space. The function is objective function from X on to d (X).

Theorem 5.4 [8]

Suppose X is a metric space, Z is a metric subspace of X and S⊆Z. Then S is a connected subset of X if and only if S is a connected subset of Z.

Topological manifold

Euclidean space and their subspace R^n are the most important. The metric space R^n serve as a topological model for Euclidean space E^n, for finite dimensional vector spaces over R or C. It is natural enough that we are led to study those spaces which are locally like R^n. We will consider spaces called manifolds, defined as follows.

Definition 6.1

A manifold M of dimension n, or n-manifold is topological space with the following properties :

- M is Housdorff space.
- M is locally Euclidean of dimension n and,
- M has a countable basis of open sets.

As a matter of notion dim M is used for the dimension of, when Dim m = 0, then M is a countable space with discrete topology

Example 6.2

Define the circle $S^1 = \{z \in \mathbb{C}: |z| = 1\}$. Then for any fixed point $z \in S^1$, write it as $z = e^{2\pi i c}$ for a unique real number $0 \leq c \leq 1$, and define the map

$vz: t \to e^{2\pi i c}$.

We note that v maps the natural $I_c = (c - \frac{1}{2}, c + \frac{1}{2})$ to the neighborhood of z given by $s^1/-z$, and it is a homeomorphism. Then $\varphi z = vz|I_c^{-1}$ is a local coordinate chart near. By taking products of coordinate charts, we obtain charts for the Cartesian product of manifolds. Hence, the Cartesian product is a manifold.

Theorem 6.3 [8]

A topological manifold M is locally connected, locally compact, and a union of a countable collection of compact subsets; furthermore, it is normal and metrizable.

Riemannian manifold

In this section we introduce the notion of a Riemannian manifold (M. g). The metric g provides us with an inner product on each tangent space an can be used to measure angels and the lengths of curve in the manifold. These terms are named after the German mathematician Bernhard Riemann. This defines a distance function and turns the manifold into a metric space in a natural way. Let M be a smooth manifold, $C^\infty(M)$ denote the commutative ring of smooth function on M and $C^\infty(TM)$ be the set of smooth vector fields on M forming a module over $C^\infty(M)$. Put $C^\infty(TM) = C^\infty(M)$ and for each positive integer $r \in Z^+$ let $C^\infty(TM) = C^\infty(M) \otimes \ldots \otimes \ldots \otimes C^\infty(TM)$ be the r-fold tensor product of $C^\infty(TM)$ over $C^\infty(M)$.

Definition 7.1

- A Riemannian manifold is a pair (M. g) consisting of a smooth manifold M and a metric g on the tangent bundle, i.e., a smooth, symmetric positive definite (0,2) -tensor field on M. The tensor g is called a Riemannian metric on M.
- Two Riemannian manifolds (Mi, Mi), (i = 1,2) are said to be isometric if there exists a diffeomorphism Φ :

$M_1 \to M_2$ such that $\Phi^* g_2 = g_1$.

Examples 7. 2

(The Euclidean space): The space R^n has a natural metric

$g0 = (dx^1)^2 + \cdots + (dx^n)^2$.

The geometry of (R^n, g_0) is the classical Euclidean geometry.

(The hyperbolic plane): The Poincare model of the hyperbolic plan is the Riemannian manifold (D, g) where D is the unit open disk in the plane R^2 and the metric g is given by

$$g = \frac{1}{1 - x^2 - y^2}(dx^2 + dy^2).$$

Theorem: (Fundamental Theorem of Riemannian geometry) 7.3

Let M be a Riemannian manifold, there exists a uniquely determined Riemannian connection on M.

Theorem 7.4 [8]

A connected Riemannian manifold is a metric space with the metric d(p, q) = infimum of the lengths of curves of class C^1 from p to q its metric space topology and manifold topology agree.

Roots systems

Roots (Introduction) 8.1

The Root or Root vectors of Lie algebra are the weigh vectors of its ad joint representation. Roots are very important because they can be used both to define Lie algebra and to build their representations We will see that Dynkin Diagrams are in fact really only away to encode information about roots. The number of Roots is equal to the dimension of Lie algebra which is also equal to the dimension of the ad joint representation, therefore we can associate a Root to every element of the algebra. The most important things about Roots is that they allow us to move from one weight to another. (weights are vectors which contain the eigenvalues of elements of Cartan sub algebra).

Definition (Root systems) 8.2

Let V be a real finite - dimensional vectors space and R ⊂ Va finite set of nonzero vectors, R is called a root systems in V (and its members called roots) if

- R generates V.
- For each ∝∈ R there exists a reflection S∝ along ∝ leavingR invariant.
- For all ∝, β ∈ R the number a$\mathcal{B}$,∝ determined by

$s_{\propto}\beta = \beta - a\mathcal{B},\propto \propto$ is an integer that is a$\mathcal{B}$,∝ ∈ z.

Theorem 8.3 [9]

Every root system has a set of simple roots such that for each α ∈ Φmay be written as

$$\alpha = \sum_{\delta\in\Delta} k_\delta \delta, \qquad 2$$

With K∂ ∈ zand each K∂ has same sign.

Example 8.4

For convenience, we introduce the roots system$B_2$by way of providing an uncomplicated example for future reference. Note that α and β as labeled form a base for B_2.

The roots which are part of a given basis are called simple. It follows from the simple roots' status as a basis that the rank of the base, i.e. the number of simple roots, is equal to the dimension of the Euclidean space E. The existence of such a base for any given root system may be proven in such a way that to determine an algorithm for finding a base given a root system. Let a root in ϕ be called indecomposable if it may not be written as a linear combination of any other roots. By selecting all the indecomposable roots whose inner product with a predetermined vector y in E is positive, one obtains a set of linearly independent roots α which lie entirely on the same side of the hyper plane normal to y. Then −α is not contained in the set for all α, and in fact these roots both span E and give rise to all other roots.

Definition (Reduced Root systems) 8.5

Let V be an Euclidean vector space (finite –dimensional real vector space with the canonical inner product

(. , .). Then R ⊆ V \ {0}is a reduced root systems if it has the following properties :

- The set R is finite and it ontains a basis of the vector space V.
- For roots α, β ∈ R we demand nα,β to be integer :

  $n_{\alpha,\beta} \equiv \frac{2(\alpha,\beta)}{(\beta,\beta)} \in Z.$
- If sα: V ⟶ V, $S_\alpha(\lambda) = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha \in Z.$
- If α, cα ∈ R for some real c,thenc = 1 or − 1

Remark 8.6

If ∝, β∈ R are porportional, β = m ∝ (m ∈ R) then $m = \pm\frac{1}{2}, \pm 1, \pm 2$ infact the numbers $a_{\propto,m_\propto} = 2$ /m and $a_{m\propto,\propto} = 2$ are both integers A root system R is said to be reduced if ∝,β ∈ R, β = m ∝ implies m = ±1.

A root ∝∈ R is called indivisible if 1/2 ∝∉ R and unmultipliable if 2∝∉ R.

Examples 8.7

- The set Δ= Δ (g, 𝔥𝔥,)of root asemisimpleLie algebra g over Cwith respect to a carton subalgebra 𝔥𝔥 is a reduced root system
- The set ∑ restricted roots is a root system which in general is not reduced.

Theorem 8.8 [9]

- Each root system has a basis.
- Any tow bases are conjugate under a unique weyl group element.
- aB,∝ ≤ 0 for any two different element ∝, $\mathcal{B}^{/}$, in the same basis.

Symmetric spaces

Symmetric spaces are of great importance for several branches of mathematics. Any symmetric space has its own special geometry, such as Euclidean, elliptic & hyperbolic geometry etc….

We can consider symmetric spaces from different points of view. In this paper, we consider their algebraic features by considering Lie groups and their Lie Algebras as algebraic approach to symmetric spaces. In fact a symmetric space can be considered as a

Lie group G with a certain involution σ, or a homogeneous space $\frac{G}{H}$ where G is a Lie group and H its isotropy subgroup. In the above sections we have discussed the important features and properties of Lie groups and their Lie algebras which help in disclosing some algebraic features of symmetric spaces. Also in this paper we cannot discuss all features such as types of symmetric spaces and their classification, but we gave introductory notions which help in future work in this field.

Involutive automorphism 9.1

Let g be a Lie algebra, the linear automorphism σ: g→ g is called an involutive automorphism if it satisfies σ^2 =I (the identity) but σ ≠ I, that is σ has eigenvalues ±1 and it splits the algebra g into orthogonal subspaces corresponding to these eigenvalues.

Symmetric Sub algebra 9.2

If g is a compact simple Lie algebra, σ is an involutive automorphism of g and g = $\mathfrak{h}\oplus$P satisfying σ (X) =X for X ∈ $\mathfrak{h}$, σ (X) = −X for X ∈ P, $\mathfrak{h}$ is a sub algebra, but P is not, and the following relations hold : [$\mathfrak{h}$, $\mathfrak{h}$] ⊂ $\mathfrak{h}$, [$\mathfrak{h}$, P] ⊂P, [P, P] ⊂ $\mathfrak{h}$ (9,2).

A sub algebra $\mathfrak{h}$ satisfying (9,2) is called symmetric sub algebra.

Cartan decomposition and Symmetric Spaces 9.3

Using what is known as Weyl unitary trick, that is by multiplying the elements in P by i we get a new noncompact algebra g* = $\mathfrak{h}$ ⊕iP, this is called a Cartan decomposition, and $\mathfrak{h}$ is a maximal compact sub algebra of g*. The Lie groups corresponding to the Lie algebras g & g* are G and H the isotropy subgroup of the Lie group G. Generally the coset space G/H is the set of subsets of G of the form gH, for g ∈ G, G acts on this coset space, that is the symmetric space.

Theorem 9.4 [9]

Any symmetric space S determines a Cartan decomposition on the Lie algebra of Killing fields. Vise versa, to any Lie algebra g with Cartan decomposition g = $\mathfrak{h}$ ⊕P there exists a unique simply connected symmetric space S = G/H where G is the simply connected Lie group with Lie algebra g and H the connected subgroup with Lie algebra $\mathfrak{h}$.

Example 9.5

Let G = S ∪ (n, C) be the group of unitary complex matrices with determinant +1. The algebra g= SU (n, ℂ) of this Lie group consists of complex antihermitian matrices of zero trace. X ∈ g Can be written as X = A + iB where A is real skew – symmetric and traceless, and B is real, symmetric and traceless. Therefore g = $\mathfrak{h}$ ⊕P where $\mathfrak{h}$ is the compact connected sub algebra SO (n, R) consisting of real, skew – symmetric and traceless matrices, and P is the subspace of matrices of the form iB, where B is real, symmetric and traceless. P is not a sub algebra. g* = $\mathfrak{h}$ ⊕ iP where iP is the subspace of real, symmetric and traceless matrices B. The Lie algebra g* = SL (n, R) is the set of n × n real matrices of zero trace and generates the linear group of transformations represented by real n × n matrices of unit determinant. The involutive automorphism that splits the algebra g is defined by the complex conjugation σ= K, and for g* the involutive automorphism is defined by σ = $(g^t)^{-1}$ for g∈g*. The decomposition g*= $\mathfrak{h}$ ⊕iP is the usual decomposition of a SL(n,R) matrix in symmetric and skew–symmetric parts Now $\frac{G}{H}$ = SU (n, ℂ) / SO (n, R)

is a symmetric space of compact type, and the related symmetric space of non – compact type is

G*/H = SL (n, R) /SO (n, R) .

In this manner, we can speak about different types of symmetric spaces especially for groups of matrices which has many applications.

More features

Here we gave some notions of algebraic and geometric features of symmetric spaces. In fact a symmetric space is a Riemannian manifold in which the geodesic symmetry at each point is an isometry in a normal neighborhood of the point (Local property). Symmetric spaces are locally symmetric where the geodesic symmetries are global isometries.

Definition 10.1 (The rank)

The rank of a symmetric space M is the dimension of the largest abelian subalgebra of P where g = $\mathfrak{h}$⊕ P.

Theorem 10.2 [9]

A complete, locally symmetric, simply connected Riemannian manifold is a symmetric space.

Examples 10.3

The Euclidean n- Space E^n, The n- sphere S^n and the hyperbolic space H^n are standard examples of symmetric spaces, also these examples can be used for introducing more symmetric spaces and their properties.

Real forms in Symmetric spaces 10.4

Real forms can be classified according to all the involutive auto morphisms of the Lie algebra g, satisfying σ^2 =I. We have two distinctive real forms which are the normal real form and the compact real form.

The normal real form of the algebra gc which is also the least compact real form, consists of the subspaces containing real coefficients c^i & c^α. It has a metric with respect the bases {Hi, E±α}. The compact real form of gc is obtained by the Weyl unitary trick:

$$\mathfrak{h} = \left\{\frac{(E_\alpha - E_{-\alpha})}{\sqrt{2}}\right\}. P = \left\{ iH_i, \frac{i\,(E_\alpha - E_{-\alpha})}{\sqrt{2}}\right\} (10,1)$$

All real forms of any complex Lie algebra can be classified with characters lying between the character of the normal real form and the compact real form. This can be done just by enumerating all the involutive automorphisms of its compact real form. If g is the compact real form of a complex semi simple Lie algebragc g^* runs through all its associated non compact real forms g^*, g' ∗ , . .. with corresponding maximal compact subgroups ђ, ђ' and complementary subspaces iP, i p', ...as σ runs through all the involutive auto morphisms of g. Also a complex algebra and all its real forms (the compact and all non-compact ones) correspond to the same root lattice and Dynkin diagram.

$G^*/H = SL(n, R) / SO(n, R)$

Example 10.5

The normal real form of the complex algebra gc = SL (n, C) is the non-compact algebra g^* = SL (n, R). This algebra can be decomposed as ђ ⊕ iP where ђ is the algebra consisting of real, skew – symmetric and traceless n × n matrices and iP is the algebra consisting of real, symmetric and traceless n × n matrices. Using the Weyl unitary trick, this algebra form the compact real form of gc, Su (n, C) = g = ђ ⊕ iP.

Applying some involutive automorphisms to the elements of the compact real form g, we can construct all the various non-compact real forms g*, g'*.

Weyl group

The Weyl group W of a root system consists of all the reflections σ_αGenerated by elements α of the root system. For a given root α, the reflection σ_α fixes the hyper plane normal to α and maps α → −α.we may write $\sigma_\alpha(\beta) = \beta - \langle\alpha, \beta\rangle \alpha$. The hyper planes fixed by the elements of W partition E into Weyl chambers. For a given base Δ of E, the unique Weyl chamber containing all vectors γ such that (γ, α) ≥ 0∀ α ∈Δ is called the fundamental Weyl chamber. We first prove the statement for W/, the subgroup of W generated by only those rotations arising from the simple roots of a given all vectors γ such that (γ, α) ≥ 0∀α ∈Δ is called the fundamental Weyl chamber.

Theorem 11.1 [14]

Given Δ andΔ/bases of a root system Φ, Δ/σ (Δ) for some σ ∈W/.

Lemma 11.2 [11]

For all α ∈ ϕ, ∃σ ∈ W such that σ (α) ∈ Δ.

Lemma 11.3 [11]

W/= { σαarising from α ∈Δ} generates W.

The cartan matrix

For a root system ϕ = αi, αj,.. one may define a matrix C by C_{ij} = ⟨αi, αj⟩.This is the Cartan matrix of ϕ. Clearly, the Cartanmatrix is not symmetric however, Cartan matrices do possess several immediately observable and distinctive features. For example the main diagonal always consists of 2's, and off-diagonal entries are restricted to integers of absolute value ≤ 3.

Definition (The generalized Cartan matrix) 12.1

A generalized Cartan matrix A = Aij is asquare matrix with integral entries such that.

- For non-diagonal entries, Aij ≤ 0.
- Aij =0 if and only if Aji=0
- A can be written as DS where D is a diagonal matrix, and S is a symmetricmatrix.

Theorem 12.2 (Cartan's first criterion) [9]

A Lie algebra g is solvable if and only if k (x y,) = 0 for all x ∈[g, g], y ∈ g.

We use Cartan's second criterion to determine if a Lie algebra is semi simple or not as follows.

Theorem 12.3 (Cartan's second criterion) [9]

A Lie algebra g is semi simple if and only if its killing form is nondegenerate.

When we study the structure of a Lie algebra, its solvability and simplicity are helpful in this field, and also when we can decompose the Lie algebra into simple ideals or using its semi simplicity.

Example 12.4

The Cartan matrix for the root system B introduced previously, has the following form$\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$.

Characterization of ϕ 12.5

As a consequence of the transitive action of the Weylgroup on bases, it may be shown that the Cartan matrix of arootsystem ϕ isindependent of the base chosen.

Theorem 12 [14]

Given two root systems ϕ⊂ E and ϕ/⊂ E/with bases

Δ = {$\alpha_i, \alpha_j, \dots, \alpha_l$} and Δ/= { α_i/ , α_j/ α_j , ... , α_l/ }with identical cartan matrices i.e ⟨α_i, α_j⟩ = ⟨ α_i/ , α_j/ ⟩ for 1≤ i, j ≤ 1.

Then this bijection extends to an isomorphism f: E ⟶ E . which maps

ϕ ⟶ ϕ/ and satisfies ⟨f(α), f (β)⟩ = ⟨α, β⟩ ∀ α, β ∈ ϕ.

Dynkin diagrams

As mentioned previously, irreducible root systems provide a simple means of classifying Lie algebras. However, the root systems may themselvesbe classified according to their Dynkin diagrams. Each such diagrams belongs to one of finitely many families of graphs with a variety of connections to e.g. quiver representations. This correspondence between Cartan matrices and Dynkin diagrams may be explicitly under stood as follows. Each vertex of the Dynkin diagram corresponds to a root αi. Clearly if $C_{ij}=C_{ji}= 0$, no edge exists between the vertices for αi and αj. If theCijth and Cjith entries in the Cartan matrix are both ±1, a single edge connects the vertices corresponding to α_i and α_j.If the C_{ij}thor C_{ji}th entry is ±2 or ±3, two or three edges, respectively, connect the two vertices in question. In order to distinguish the relative lengths of the roots, an arrow pointing to wards the shorter of the two is drawn over the vertex in question. The properties of the Cartan matrices place a number of restrictions on possible. Dynkin diagrams, which we enumerate below. Intact, these properties, enumerated below, lead to a complete description of all possible Dynkin diagrams, which may be found in

- If some of the vertices of the Dynkin diagram are omitted along with all their attached edges, the remaining graph is also possible as a Dynkin diagram.
- The number of vertex pairs connected by at least one edge is strictly less than the order of the root system. It follows that no Dynkin diagram may contain a cycle.
- No more than three edges can connect to a single vertex. Thus, the only Dynkin diagrams containing a triple edge contain exactly those two vertices it connects.
- If a Dynkin diagram contains as a sub graph a simple chain, the graph obtained by reducing that chain to a point also forms a Dynkin diagram. This prohibits several possible arrangements of terminal vertices from co-occurring within a diagram, lest the preceding restriction be violated.

Definition (Dynkin diagram) 13.1:

Suppose s ⊆ R is a simple root system. The Dynkin diagram of s is a graph constructed by the following prescription

- For each αi, ∈ S we construct a vertex (visually, we draw a circle).
- For each pair of roots αi,,αj, we draw a connection depending on the angle φ between them.
- If φ = 90°the vertices are not connected (we draw no line).
- If φ = 120° the vertices have a single edge (we draw a single line).
- If φ = 135° the vertices have a double edge (we draw two connecting lines).
- If φ = 150 the vertices have a triple edge (we draw a three single connecting lines).

For double and triple edge connecting two roots, we direct them towards the shorter root (we draw an arrow pointing to the shorter root).

Example 13.2

The Dynkin diagram for our familiar root system, B_2, is asfollows. Recall that e.g. βis longer than α: B2.

Example 13.3

The only Dynkin diagram with a triple edge, C_2, has the following form: C2

Main Results

- The elements of a Lie group can act as transformations on the elements of the symmetric space.
- If M is a symmetric space, its group of isometries G has a Lie group structure and we can obtain all information of M from G. If the point p∈ M, H the isotropy subgroup at p and g is the Lie algebra of G, then the Lie algebra ɧ of H is a subalgebra of g having a complementary subspace P such that g = ɧ ⊕ P, [ɧ, ɧ] ⊂ ɧ, [ɧ, P] ⊂ P and [P, P] ⊂ P, and so the triple (g, ɧ, P) gives characterization of symmetric
- Every Lie algebra corresponds to a given root system, and each symmetric space corresponds to a restricted root system.
- We can have several different spaces derived from the same Lie algebra.
- Most of features of symmetric spaces can be extracted from Lie algebras.
- Many properties of symmetric spaces can be studied through their Lie algebras and root systems.

Conclusion

- In this scientific paper, the importance of Root Systems, Cartan Matrix, Dynkin Diagrams in Classification of Symmetric Spaces.

Bibliography

1. Dyson F. *Communications in Mathematical Physics* 19 (1970): 235.
2. Olshanetsky MA and Perelomov AM. *Physical Report* 94 (1983): 313.
3. Hermann R. "Lie Groups for physicists". (W.A Benjamin Inc. , New York) (1966).
4. Ulrika Magnea. "An introduction to symmetric spaces". Dept. Of Math university of Torino 10,1-10125, Italy.
5. Stephanie Kernik. "A very brief overview of lie algebras". University of Minnesota, Morris, (2008).
6. Humphreys James E. "Introduction to Lie algebras and representation theory". Springer, (1994).
7. Sattinger DH and weaver OL. "Lie Group and Lie algebras with application to physics, geometry and mechanics". Springer – verlag, new York (1986).
8. Boothby WM. "An Introduction to differentiable Manifolds and Riemannian Geometry". Academic Press, New York (1975).
9. Helgason S. "Differential geometry, Lie Groups and symmetric spaces". Academic press, New York (1978).

10. Gilmore R. "Lie Groups, lie algebras and Some of Their Applications". John wiley and Sons, New York (1974).

11. M Brion. "Representations of Quivers". Unpublished lecture notes (2008).

12. H Derksen and J Weyman. "Quiver Representations". Notices of the AMS 52.2 (2005).

13. P Etingof., *et al*. "Introduction to Representation Theory". Unpublished lecture notes. (2009).

14. J E Humphreys. "Introduction to Lie Algebras and Representation Theory". Springer-Verlag. 1972 Theory. Springer. (1983).

15. V G Kac. "Root Systems, Representations of Quivers and Invariant Theory". Springer (1983).