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Review Article

Medial i-quasigroups and Tarski quasigroups

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Abstract

We research medial i-quasigroups and Tarski quasigroups.

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Introduction

I. A. Florja and N. N. Didurik researched i-quasigroups [3].

Definition 1

Quasigroup with i-identity

$x (xy \cdot z) = y (zx \cdot x)$.....(1)

is called i-quasigroup.

This identity is similar to Moufang identity [1,6,9] but it is not the same identity. Quasigroups with this identity form a new quasigroup class [3]. Here we study medial i-quasigroups and i-T-quasigroups.

Necessary definitions can be found in [1,6,9].

Definition 2

Binary groupoid $(Q, \circ)$ is called a left quasigroup if for any ordered pair $(a, b) \in$

Q2 there exist the unique solution $x \in Q$ to the equation $a \circ x = b$ [1].

Definition 3

Binary groupoid $(Q, \circ)$ is called a right quasigroup if for any ordered pair $(a, b) \in$

Q^2 there exist the unique solution $y \in Q$ to the equation $y \circ a = b$ [1].

Definition 4

Binary groupoid (Q,.) is called medial if this groupoid satisfies the following medial identity:

for all $x, y, u, v \in Q$ [1].

$xy \cdot uv = xu \cdot yv$.......... (2)

We recall

Definition 5

Quasigroup $(Q, \cdot)$ is a T-quasigroup if and only if there exists an abelian group $(Q, +)$, its automorphisms ϕ and ψ and a fixed element $a \in Q$ such that $x \cdot y = \phi x + \psi y + a$ for all $x, y \in Q$ [5].

A T-quasigroup with the additional condition $\phi\psi = \psi\phi$ is medial quasigroup.

i-identity in medial quasigroups

Examples of i-quasigroups (including and i-quasigroups of order 5) are given in [3].

Theorem 1

In medial quasigroup (Q,) of the form x . y = ϕx + ψy i-identity is true if and only if $\psi = \phi^2$, $\phi^4 = \varepsilon$.

Proof. We rewrite identity (1) in the following form:

φx + ψ (ϕ (ϕx + ψy) + ψz) = ϕy + ψ (ϕ (ϕz + ψx) + ψx), ………. (3)

If we substitute in equality (3) x = y = 0 then we have

$\psi = \varphi^2$-----------(4)

If we substitute in equality (3) x = z = 0 then we have

ψφψ = φ-------(5)

and taking into consideration that ϕψ = ψϕ we have

$\psi^2 = \varepsilon$ ----------- (6)

Therefore, we can rewrite previous equalities in the form

$\psi^2 = \varphi 4 = \varepsilon$ ---------- (7)

If we substitute in equality (3) y = z = 0 then we have

$\varphi x + \psi\varphi^2 x = \psi\varphi\psi x + \psi^2 x$ ------- (8)

Equality (8) follows from equalities (7).

Converse. If we substitute in identity (1) the expression x y = ϕx + ψy, then we obtain equality (3), which is true taking into consideration equality (7). Then we obtain, that identity (1) is true in this case.

See examples of i-quasigroups from [3].

Suppose that we have a cyclic group Z_n. It is well known [4] that Aut (Z_n) = Z_n^* is a cyclic group and $|Z_n^*|$ = ϕ (n), where ϕ (n) is Euler function:

$$\phi(n) = n \prod_{p|n} (1 - 1/p), n > 1$$

Here p is prime number, and p runs through all the values involved in the decomposition of n into prime factors [2].

Notice, ϕ (mn) = ϕ (m)·ϕ (n), if the numbers m and n are mutually prime numbers (coprime), and $\phi(p^k) = p^k - p^{k-1}$, if the number p is a prime number [4].

Example 1

Suppose we have an abelian (commutative) group with automorphisms ϕ = ψ = ε. In such group i-identity is true.

Theorem 2

Suppose that we have commutative (abelian) group C such that the group Aut (C) contains an element ϕ of order 4. Then medial quasigroup of the form x y = $\phi x + \phi^2 y$ is medial i-quasigroup.

Proof. We can rewrite i-identity x (xy · z) = y (zx · x) in the form

$\varphi x + \varphi^2 (\phi (\phi x + \phi^2 y) + \phi^2 z) = \phi y + \phi^2 (\phi (\phi z + \phi^2 x) + \phi^2 x)$,

ϕx + x + ϕy + z = ϕy + z + ϕx + x, 0 = 0.

Example 2

We have ϕ (10) = 4.

Let's consider quasigroup x y = 3. x-y (mod 10). We check that this is medial i-quasigroup. We can rewrite i-identity x (xy . z) = y (zx . x) in the form

3x - (3 (3x - y) - z) = 3y - (3 (3z - x) - x) (mod 10),

3x - 9x + 3y + z = 3y - 9z + 3x + x (mod 10),

3x - 9x + 3y + z = 3y - 9z + 3x + x (mod 10),

-6x + z = -9z + 4x (mod 10),

10z = 10x (mod 10),

0 = 0 (mod 10).

Example 3

We have ϕ (15) = 8. Let's consider quasigroup x ∘ y = 2. x + 4. y (mod 15). We check that this is medial i-quasigroup. We can rewrite i-identity x (xy. z) = y (zx. x) in the form

2x + 4 (2 (2x + 4y) + 4z) = 2y + 4 (2 (2z + 4x) + 4x) (mod 15),

0 = 0 (mod 15).

Example 4

We can present elements of the group (Z_2^3, +) in the following form : 1 = (000), 2 = (001), 3 = (010), 4 = (011), 5 = (100), 6 = (101), 7 = (110), 8 = (111).

+	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	6	5	8	7
3	3	4	1	2	7	8	5	6
4	4	3	2	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	8	7	2	1	4	3
7	7	8	5	6	3	4	1	2

8	8	7	6	5	4	3	2	1

We start from the groups GL(3; 2) 3 ∼= PSL(2; 7). This is the group of non-degenerate matrices of size 3 × 3 over the field of order 2. Or the group of non-degenerate matrices of size 2 × 2 over the field of order 7.

= |GL (3, 2)| = 168 = 3 × 7 × 8 [4].

$$\varphi = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \psi = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

isms of the group $(Z_2^3, +)$:

Notice that $\phi^2 = \psi$, $\phi^4 = \psi^2 = \varepsilon$, $\phi\psi = \psi\phi$ since $\phi\phi^2 = \phi^2\phi$.

We see that.

Φ (000) = ϕ (1) = (000) = 1; ϕ (100) = ϕ (5) = (010) = 3; ϕ (010) = ϕ (3) = (100) = 5; ϕ (001) = ϕ (2) = (101) = 6; ϕ (011) = ϕ (4) = (001) = 2; ϕ (110) = ϕ (7) = (110) = 7; ϕ (101) = ϕ(6) = (111) = 8; ϕ(111) = ϕ(8) = (011) = 4.

Ψ (000) = ψ (1) = (000) = 1; ψ (100) == ψ (5) = (100) = 5; ψ (010) = ψ (3) = (010) = 3; ψ (001) = ψ (2) = (111) = 8; ψ (110) = ψ (7) = (110) = 7; ψ (011) = ψ (4) = (101) = 6; ψ (101) = ψ (6) = (011) = 4; ψ (111) = ψ (8) = (001) = 2.

∘	1	2	3	4	5	6	7	8
1	1	8	3	6	5	4	7	2
2	6	3	8	1	2	7	4	5
3	5	4	7	2	1	8	3	6
4	2	7	4	5	6	3	8	1
5	3	6	1	8	7	2	5	4
6	8	1	6	3	4	5	2	7
7	7	2	5	4	3	6	1	8
8	4	5	2	7	8	1	6	3

quasigroup (Q, ∘) has the form $x \circ y = \phi x + \psi y$

Tarski identity in medial quasigroups

There exist at least two identities that have name Tarski identity a (b (ca)) = cb [8] and identity (x(zy) = (xy)z) [7]. Identity x (zy) = (xy) z in details is researched in [7].

Here we concentrate over medial quasigroups with the following Tarski identity [8]:

X (y (zx)) = zy--------(10)

Theorem 3

In medial quasigroup (Q,) of the form x y = ϕx + ψy Tarski identity is true if and only if ϕ = ε, ψ = I.

Proof. We rewrite identity (10) in the following form.

φx + ψ (ϕy + ψ (ϕz + ψx)) = ϕz + ψy------- (11)

If we substitute in equality (11) x = y = 0 then we have

$\psi^2\varphi z = \varphi z$, $\psi^2 = \varepsilon$.----- (12)

If we substitute in equality (11) x = z = 0 then we have

ψφy = ψy, φ = ε.------- (13)

If we substitute in equality (11) y = z = 0 then we have

$\varphi x + \psi^3 x = 0$, φx + ψx = 0. --------- (14)

Therefore, we can rewrite previous equalities in the form.

φ = ε, ψ = I, --------- (15)

where x - Ix = 0 for all x ∈ Q.

Converse

If we substitute in identity (10) the expression x. y = φx + ψy, then we obtain equality (11), which is true taking into consider-

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