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Research Article

Low-pass Equivalent Dynamics and Control in Systems with Nonlinear Coupling of Linear Oscillators

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Abstract

Control theory of nonlinear systems receives continuously increasing attention. System nonlinearity occurs when at least one subsystem is nonlinear. Classical methods used for linear systems, particularly superposition, are not applicable to the nonlinear systems. It is necessary to use other methods to study the control of these systems. For a wide class of nonlinear systems, a rather important feature is the strong coupling nonlinearity between spectrally decoupled parts. Even in the case of low frequencies, where lumped models can still be employed the nonlinear coupling between parts of the system requires specific treatment, using advanced mathematical tools. A frequency domain approach is employed for systems with linearly decoupled but nonlinearly coupled subsystems. The Hilbert transform is appropriately introduced for obtaining two low-pass subsystems that form an equivalent description of the essential overall system dynamics. The nonlinear coupled dynamics is investigated systematically by partitioning the coupled system state vector in such a way as to fully exploit the low-pass and the band-pass intrinsic features of free dynamics. In particular, by employing the Hilbert Transform, a low-pass equivalent system is derived. Then, a typical case is investigated via numerical simulation of the original coupled low and band-pass, real-state-variable system and the low-pass-equivalent, complex-state-variable derived one. The nonlinear model equations considered here enable a systematic investigation of nonlinear feedback control options to operate mechatronic transducers in energy harvesting, sensing or actuation modes.

Keywords: Dynamics and Control; Feedback Linearization; Mechatronics; MEMS; Nonlinear Control; Nonlinear Systems

State-Space Modulation and Demodulation

Introduction

The analysis of coupled dynamic systems including nonlinearly coupled but spectrally separated and effectively linearly decoupled, as demonstrated later on, subsystems has gained significant attention recently. This is due, at least partly, to the developments and associated needs in the disciplines of telecommunications and mechatronics, e.g. [1-6]. In Figure 1 a generic model of a communication channel as a distributed parameter electromagnetic system is shown.

However, the most important and rather common feature of all widely used channels is their frequency selectivity. Indeed, the bands where transmission with relatively low attenuation is possible are limited both in number as well as in extent. Therefore, modulation of the carrier wave by the information signal must be employed. Modulation is practically relocation of the information signal spectral content to a band which is mostly suited for information transmission over a specified channel. The simpler, yet the most usual, modulation method is amplitude modulation.

In the wider context of mechatronics, electromechanical systems analysis and synthesis, especially in comparatively small scales, has gained significant interest, due to developments in the rapidly evolving field of MEMS/NEMS (Micro-Electro-Mechanical Systems, Nano-Electro-Mechanical Systems) [1]. The final objective is to develop integrated devices that can be used as tiny actuators or sensors, which, due to low manufacturing cost, can be used in large quantities for monitoring and control of complex systems and processes, like e.g. structural health monitoring of aircraft, spacecraft, watercraft, automobiles etc., as well as industrial plant monitoring and control. In the typical case, a small-scale mechanical system like a resonator or a spring-mass-damper interconnection is driven by analogue electronics, like voltage or current sources, amplifiers etc., which, in turn, are accurately controlled by digital embedded components like DSPs (Digital Signal Processors) etc. that actually implement the intelligence in the device. Parameters like silicon chip surface utilization, power consumption and level of integration are of utmost important for commercial and technological success of a newly proposed design.

Nonetheless, a rather important feature in the analysis of such systems comes from the strong nonlinearity commonly appearing in the coupling between spectrally decoupled parts of the system. As seen in the analysis, even in the case of low frequencies, where lumped models can still be employed the nonlinear coupling between parts of the system requires specific treatment, using advanced mathematical tools [1,2].

In this context, an alternative, frequency-domain state-space approach is pursued here. In the rest of this work, a specific class of systems with structure comprising linearly decoupled but nonlinearly coupled subsystems is examined. The mathematical toolbox of the Hilbert transform is appropriately introduced for obtaining two low-pass subsystems that form an equivalent description of the essential overall system dynamics. The procedure is then applied to an arrangement commonly encountered in mechatronics and energy harvesting. In this arrangement, a voltage or current source is coupled to a mechanical second-order oscillator, consisted of a mass-damper-spring interconnection, through an electromagnet [1,3-7]. Such an arrangement is shown in figure 2. A voltage source is driving the RLC circuit which is coupled to the mechanical oscillator through the inductor's armature. In this configuration, the voltage source plays a dual role: it is a source of power that allows the mechanical part of the system to move and, at the same time, of the information control signal that drives the mass (payload) of the mechanical subsystem to the desired position. Furthermore, the electromechanical coupling plays a dual role as well. Indeed, a mutual interaction is established. The movement of the metallic mass induces a change to the value of the inductance apparent to the rest of the driving electric circuit. In this system the low-pass part is the mechanical oscillator and the band-pass part is the electric circuit that drives the electromagnet.

Figure 1: Electromagnetic telecommunication channel model.

Figure 2: Coupled electromechanical oscillator.

The general system form in state space

As demonstrated later in a detailed example, a general class of dynamic systems will be considered. The description of this class of systems is cast in a state-space framework. Let vector **x** denote the full state vector of the coupled system. Based on a properly defined partitioning of the coupled system's state vector, this description admits the following mathematical structure:

$$\dot{x}_1 = A_{LP} \cdot x_1 + \psi(y_2) + d \qquad (1.1)$$

$$\dot{x}_2 = F(x_1) \cdot x_2 + G(x_1) \cdot u$$

$$y_2 = C_2 \cdot x_2 \qquad (1.2)$$

The various terms entering the above equations are defined as follows:

$x_1, d \in \mathbb{R}^{n_1}, A_{LP} \in \mathbb{R}^{n_1 \times n_1}$

$x_2 \in \mathbb{R}^{n_2}, u \in \mathbb{R}^{m_2}, y_2 \in \mathbb{R}^{p_2}, C_2 \in \mathbb{R}^{p_2 \times n_2}$

$\psi: y_2 \in \mathbb{R}^{p_2} \to \psi(y_2) \in \mathbb{R}^{n_2}$

$F: x_1 \in \mathbb{R}^{n_1} \to F(x_1) \in \mathbb{R}^{n_2 \times n_2}$

$G: x_1 \in \mathbb{R}^{n_1} \to G(x_1) \in \mathbb{R}^{n_2 \times m_2}$

The partitioning of the n-dimensional state vector **x** into two components $\mathbf{x}_1$ (dimension n_1) and $\mathbf{x}_2$ (dimension n_2) respectively is reflecting the partitioning of the system to a low-pass (LP) and a band-pass (BP) part. This is made clearer by elaborating on the motivation leading to the introduction of LP and BP systems.

To this end, for Single-Input-Single-Output (SISO) linear, asymptotically stable systems, whose dynamics can be defined by a scalar transfer function without poles in the right-half s-plane or on the imaginary axis, the following LP system definition is given:

$$\forall \varepsilon > 0, \exists W(\varepsilon) > 0: |\omega| > W(\varepsilon) \Rightarrow |H(\omega)| < \varepsilon \qquad (1.3)$$

Real-valued function $H(\omega) = H(s = j\omega = j2\pi f)$ or the transfer function of the SISO system. The parameter BW = (2W) with W such that

$\varepsilon = \frac{\|H(\omega)\|_\infty^2}{2} = \frac{\left[\sup_{\omega \in \mathbb{R}} (H(\omega))\right]^2}{2}$ is called half-power bandwidth or 3dB bandwidth (or simply bandwidth if there is no chance for confusion) of the system.

Generalization of the LP system leads to BP ones defined as follows:

$$\exists \omega_c, \forall \varepsilon > 0, \exists W(\varepsilon) > 0: |\omega \pm \omega_c| > W(\varepsilon) \Rightarrow |H(\omega)| < \varepsilon \qquad (1.4)$$

In analogy, a bandwidth BW is defined for BP systems. Angular frequency ω_c is called the carrier frequency of the system. Evidently, LP systems are BP systems with $\omega_c = 0$.

In the case of linear, asymptotically stable, Multi-Input-Multi-Output (MIMO) systems the above definitions may be straightforward generalized, with respect to the transfer function matrix $\mathbf{H}(s)$ of the system and its frequency-dependent maximum Singular Value $\sigma_{max}\{\mathbf{H}(\omega)\}$. For example, the BP system definition may be extended in the MIMO case as follows:

$$\exists \omega_c, \forall \varepsilon > 0, \exists W(\varepsilon) > 0: |\omega \pm \omega_c| > W(\varepsilon) \Rightarrow \sigma\{H(\omega)\}_{max} \qquad (1.5)$$

Finally, it is mentioned the definitions of BP and LP systems can straightforwardly be extended to finite energy, scalar or vector signals.

To carry out the analysis of the partitioned formulation of the system, Taylor expansions around zero for the typical multivariable vector and matrix functions will be employed. In particular, we have

$$\begin{matrix}\psi \\ F(\xi) \\ G\end{matrix} = \begin{matrix}\psi_0 \\ F_0 \\ G_0\end{matrix} + \begin{matrix}\psi_1 \\ F_1 \\ G_1\end{matrix}(\xi) + \begin{matrix}\psi_2 \\ F_2 \\ G_2\end{matrix}(\xi \otimes \xi) + \begin{matrix}\psi_3 \\ F_3 \\ G_3\end{matrix}(\xi \otimes \xi \otimes \xi) + \cdots \qquad (1.6)$$

Symbol $\otimes$ denotes the Kronecker vector product (tensor product) and the shorthand notation $\xi^{\otimes k}$ will be used to denote the k-th power of vector ξ in the Kronecker product sense.

The procedure of producing the terms in the expansion above is carried out in an element-wise manner. In specific, it is demonstrated for the case of matrix function F but can be straightforwardly generalized for ψ and G. First the multi-index, α, notation is introduced to simplify the expressions.

$$\alpha \triangleq (\alpha_1, \alpha_2, \ldots, \alpha_N); |\alpha| \triangleq \alpha_1 + \alpha_2 + \cdots + \alpha_N, \alpha! \triangleq (\alpha_1!)(\alpha_2!)\cdots(\alpha_N!); \alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{N}.$$

In effect:

$$F(\xi) = [f_{ij}(\xi)], 1 \le i, j \le K; \xi \triangleq [\xi_1 \quad \xi_2 \quad \ldots \quad \xi_N]^T$$

$$\Downarrow \qquad (1.7)$$

$$F_\alpha(\xi^{\otimes \alpha}) = \left[f_{ij}^{(\alpha)} \cdot \xi^{\otimes \alpha}\right], 1 \le i, j \le K; \xi^{\otimes \alpha} \triangleq \left[\xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_N^{\alpha_N}\right]$$

In the above, the α-th derivative vector, $\mathbf{f}_{ij}^{(\acute{a})}$ is defined by the following.

$$\boldsymbol{f}^{(\alpha)} \triangleq \left[\frac{1}{\alpha!}\partial^{\alpha}\boldsymbol{f}\right];\ \partial^{\alpha}\boldsymbol{f} \triangleq \frac{\partial^{|\alpha|}\boldsymbol{f}}{\partial\xi_1^{\alpha_1}\partial\xi_2^{\alpha_2}\cdots\partial\xi_N^{\alpha_N}} \qquad (1.8)$$

One needs to match the expansion of the operator in case considered with respect to the Kronecker vector power $\mathbf{x}_1^{\otimes k}$; also, K = n_2 and N = n_1. Note that the partial derivatives, appearing in the operator expansion, are evaluated at $x_1 = 0$ and some of the elements of $\mathbf{f}_{ij}^{(k)}$ may need to be set to zero to avoid repetitions, since an underlying assumption is Clairaut's Theorem [8] according to which the order of mixed derivatives can be interchanged. Special cases are k = 0 and k = 1:

$$\left.\begin{matrix} \boldsymbol{F}_0 = \boldsymbol{F}(\boldsymbol{0}) = [f_{ij}(\boldsymbol{x}_1 = \boldsymbol{0})], \boldsymbol{x}_1{}^{\otimes 0} = [1 \quad \dots \quad 1]^T \\ \boldsymbol{F}_1(\boldsymbol{x}_1) = [\nabla f_{ij}(\boldsymbol{x}_1 = \boldsymbol{0}) \cdot \boldsymbol{x}_1], \boldsymbol{x}_1{}^{\otimes 1} = \boldsymbol{x}_1 \end{matrix}\right\}; 1 \le i,j \le n_2 \qquad (1.9)$$

Note that, as in the general case, the elements of $\mathbf{F}_1$ are scalar products of the Kronecker power vector for k = 1 times the element-wise partial derivative (grad) vector evaluated at $x_1 = 0$.

By using the expansion for matrix functions **F**, **G**, the system equations are reformed as follows

$$\dot{\boldsymbol{x}}_2 = \boldsymbol{F}_0 \cdot \boldsymbol{x}_2 + \underbrace{[\boldsymbol{G}_0 \quad \boldsymbol{\Gamma}_0]}_{\boldsymbol{\Gamma}} \cdot \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{r} \end{bmatrix}$$

$$\boldsymbol{y}_2 = \boldsymbol{C}_2 \cdot \boldsymbol{x}_2 \qquad (1.10)$$

In the above, auxiliary signal vector **r** is defined as follows:

$$\boldsymbol{r}(t) \triangleq \sum_{k=1}^{\infty}\left\{\boldsymbol{F}_k\left(\boldsymbol{x}_1{}^{\otimes k}\right) \cdot \boldsymbol{x}_2 + \boldsymbol{G}_k\left(\boldsymbol{x}_1{}^{\otimes k}\right) \cdot \boldsymbol{u}\right\} \qquad (1.11)$$

Furthermore, it holds that:

$$\boldsymbol{\Gamma} \in \mathbb{R}^{n_2 \times (m_2 + n_2)}, \boldsymbol{\Gamma}_0 \in \mathbb{R}^{n_2 \times n_2}$$

$\mathbf{\Gamma}_0$ is a square binary matrix (i.e. (0,1)-matrix) accounting for the fact that certain elements of signal vector r may be identically zero.

In the sequel, with respect to the BP segment of our original equations, systems of form (1.10) will be considered with the additional assumption that the following transfer function matrix is BP around a carrier frequency ω_c:

$$\boldsymbol{H}_2(s) = \boldsymbol{C}_2 \cdot (s\boldsymbol{I} - \boldsymbol{F}_0)^{-1} \cdot \boldsymbol{\Gamma} \qquad (1.12)$$

The coupling of such BP systems to LP systems will be investigated with the additional assumption that the following transfer function matrix is LP:

$$\boldsymbol{H}_1(s) = (s\boldsymbol{I} - \boldsymbol{A}_{LP})^{-1} \qquad (1.13)$$

The analysis is presented employing the mathematical apparatus to establish the equivalent low-pass model. This, however, requires the Hilbert Transform which is briefly presented next.

Equivalent low-pass model of a system with state-space modulation

We now return to the coupled oscillators, equations (1.1) and (1.2), by taking into account that the transfer function matrix is BP for one segment of the system while the other one LP. Then, an equivalent, exclusively LP system model may be obtained, by using the complex envelope of the BP signal vector $x_2(t)$.

At first, due to the assumption state vector $x_1(t)$ is LP, as it is generated by an LP system. Indeed, assuming zero initial conditions, the state vector is given in the frequency domain by the following relation:

$$\mathbf{x}_1(\omega) = \mathbf{H}_1(\omega) \cdot \mathbf{d}_1(\omega) \qquad (1.14)$$

In the above, the auxiliary signal $d_1(t)$ is defined as follows:

$$\mathbf{d}_1(t) = \boldsymbol{\psi}(\mathbf{y}_2(t)) + \mathbf{d}(t) \qquad (1.15)$$

Therefore, if $d_1(t)$ is assumed to be a random signal of white Gaussian noise content, i.e. possessing power spectral density constant over all frequencies, $x_1(t)$ will comply to the low-pass requirement.

The next step is to observe that an arbitrary Kronecker power expansion of an LP signal vector is also LP with possibly larger BW. Therefore, the elements of matrices F, G if viewed as scalar signals are LP, at least in the case that the expansions in equation (1.6) obtain a finite number of terms. Properties of the Hilbert Transform given in [21] guarantee that the following holds:

$$\begin{matrix} \dot{\mathbf{x}}_2 = \mathbf{F}(\mathbf{x}_1) \cdot \mathbf{x}_2 + \mathbf{G}(\mathbf{x}_1) \cdot \mathbf{u} \\ \Downarrow \\ \dot{\hat{\mathbf{x}}}_2 = \mathbf{F}(\mathbf{x}_1) \cdot \hat{\mathbf{x}}_2 + \mathbf{G}(\mathbf{x}_1) \cdot \hat{\mathbf{u}} \end{matrix} \qquad (1.16)$$

In the above the following fact for the time derivative (denoted by a dot placed above the signal) of a signal vector has been used:

$$\hat{\dot{\mathbf{x}}}(t) = \dot{\hat{\mathbf{x}}}(t) \qquad (1.17)$$

This is a direct consequence of the linearity property of the Hilbert transform. If the second equation in (1.16) is multiplied by j and then added to the first, the following dynamic equation for the pre-envelope, x_{2+}, of the state vector signal x_2 is obtained:

$$\dot{\mathbf{x}}_{2+} = \mathbf{F}(\mathbf{x}_1)\cdot\mathbf{x}_{2+} + \mathbf{G}(\mathbf{x}_1)\cdot\mathbf{u}_+ \quad (1.18)$$

Because of the assumption that the transfer function in equation (1.12) is BP with carrier frequency $\omega_c = 2\pi f_c$ the pre-envelope x_{2+}, of the state vector signal x_2 may be expressed as a product between a modulating complex envelope factor, $\tilde{\mathbf{x}}_2$, and an imaginary scalar exponential signal acting as a generalized sinusoidal carrier signal

$$x_{2+}(t) = \tilde{x}_2(t)\,exp(j\omega_c t)$$
$$\Downarrow$$
$$\dot{x}_{2+}(t) = \dot{\tilde{x}}_2(t)\,exp(j\omega_c t) + j\omega_c\tilde{x}_2(t)\,exp(j\omega_c t) \quad (1.19)$$

If the input signal vector u is assumed tuned to the BP system, then it may be expressed similarly as follows:

$$u_+(t) = \tilde{u}(t)\,exp(j\omega_c t) \quad (1.20)$$

By substituting the equations above in equation (1.18) the following relationship is finally obtained for the complex envelope of the BP system's state vector:

$$\dot{\tilde{x}}_2 = [F(x_1) - j\omega_c I]\cdot\tilde{x}_2 + G(x_1)\cdot\tilde{u} \quad (1.21)$$

If a similar expression for the output vector, $\mathbf{y}_2$ is adopted then one obtains the following expression for its complex envelope $\tilde{\mathbf{y}}_2$; indeed, if the first equation (1.19) is substituted in the output equation of the BP subsystem:

$$\left.\begin{aligned} y_{2+}(t) &= \tilde{y}_2(t)\,exp(j\omega_c t) \\ y_2(t) &= Re\{y_{2+}(t)\}\end{aligned}\right\} \Rightarrow \tilde{y}_2 = C_2\cdot\tilde{x}_2 \quad (1.22)$$

By using the above, one can rewrite output equation of the BP subsystem so that it contains the I and Q components of $\tilde{\mathbf{y}}_2$, y_{2C} and y_{2S} respectively. This is done by employing directly the expression for ψ of equation (1.6) in the expressions for in-phase (I) and quadrature (Q) component as in decomposition in [21]. However, further simplification is possible by exploiting the assumption that transfer function (1.13) is LP in order to eliminate high-frequency terms, i.e. terms including a "carrier" factor of the form $exp(\pm j\omega_c t)$, k = 1,2,3,.... Such factors appear as a direct consequence of the fact that the coupling term in the LP subsystem's dynamic equation is the nonlinear function ψ. Then, terms with factors of the form $exp(\pm j\omega_c t)$, k = 1,2,3,... may be neglected on the basis of the spectral decoupling between the LP and the BP system. This is translated to the requirement that the LP system's BW is sufficiently smaller than the BP system's carrier frequency $\omega_c = 2\pi f_c$.

By rewriting equations (1.6) and (1.7) in the case of multivariable vector function $\psi(y_2)$ one can obtain that:

$$\psi(y_2) = \psi(y_2 = 0) + \left[\sum_{k=1}^{\infty}\psi_i^{(k)}\cdot y_2^{\otimes k}\right], 1 \le i \le n_1 \quad (1.23)$$

An LP contribution from the above is possible only if k is even. Indeed, an LP contribution is this part of each term in the above expansion which is not multiplied by a carrier factor $exp(\pm j\omega_c t)$, k = 1,2,3,.... Therefore, despite the 0-th term, which obviously participates in the LP part of the signal vector in the equation above for ψ, all terms for strictly positive k contain a component multiplied by $exp(\pm j\omega_c t)$, k = 1,2,3,.... However, when k is odd only this component is present; e.g. for k = 1. On the other hand, when k is even, except the carrier-multiplied component, there exists a carrier-free one, too. This is made clearer with an example, e.g. for k = 2.

In this case:

$$y_2^{\otimes 2} = \left(Re\left[\tilde{y}_2 e^{j\omega_c t}\right]\right)^{\otimes 2} = \left(\frac{\tilde{y}_2 e^{j\omega_c t} + \tilde{y}_2^{*} e^{-j\omega_c t}}{2}\right)^{\otimes 2} =$$
$$= \frac{\tilde{y}_2 \otimes \tilde{y}_2^{*}}{2} + \left(\frac{\tilde{y}_2}{2}\right)^{\otimes 2} e^{j2\omega_c t} + \left(\frac{\tilde{y}_2^{*}}{2}\right)^{\otimes 2} e^{-j2\omega_c t} =$$
$$= \frac{y_{2C}^{\otimes 2} + y_{2S}^{\otimes 2}}{2} + \left(\frac{\tilde{y}_2}{2}\right)^{\otimes 2} e^{j2\omega_c t} + \left(\frac{\tilde{y}_2^{*}}{2}\right)^{\otimes 2} e^{-j2\omega_c t} \quad (1.24)$$

In the above, only the first term is an LP one as the other two contain a carrier factor.

In conclusion, the equations (19) and (20) of coupled oscillators, with LP and BP transfer function matrices as in equations (31) and (30), respectively, can be reduced to the following LP equivalent system of equations:

$$\dot{x}_1 = A_{LP}\cdot x_1 + \psi_{LP}(\tilde{y}_2) + d \quad (1.25)$$
$$\dot{\tilde{x}}_2 = [F(x_1) - j\omega_c I]\cdot\tilde{x}_2 + G(x_1)\cdot\tilde{u}$$
$$\tilde{y}_2 = C_2\cdot\tilde{x}_2 \quad (1.26)$$

In the above, all signals in the BP subsystem have been substituted by their LP complex envelopes, e.g. $x_2(t)$ by $\tilde{x}_2(t) = x_{2C}(t) + jx_{2S}(t)$. Furthermore, in the LP subsystem equation, multivariable vector function $\psi(y_2)$ has been substituted by the LP one $\psi_{LP}(\tilde{y}_2) = \psi_{LP}(y_{2C}, y_{2S})$, which is produced by omitting from the original all odd terms and the modulated carrier parts of the even terms. The main benefit in using the description of equations (1.25) and (1.26) instead of the original ones in equations (1.1) and (1.2) is that, because it is LP but otherwise grasps all the essential dynamics of the dynamical system at hand, the time step for the integration of the dynamic equations may be set to a substantially smaller value than in the original one. For example, in typical mechatronic applications, as the one presented as example later in this text, the BW of both the LP and the BP system is commonly in the order of magnitude of 10 Hz. However, the carrier frequency may be 1 kHz or even higher. Therefore, the integration step may be increased at least two orders of magnitude; such a possibility makes investigations using numerical simulation much easier. Another benefit is that by using the equivalent LP system the carrier frequency, which does not play such a crucial role in the understanding of the dynamics, comes into the analysis simply as a selectable parameter. In effect, as far as the main assumptions are satisfied the selection of the carrier frequency does not affect any significant conclusions for the behavior of the system at hand.

Exact linearization of modulated state systems

An alternative problem formulation

We return now to the system formulation considered earlier, that exhibits nonlinear cross-band coupling and state modulation and demodulation.

$$\dot{x}_1 = A_{LP} \cdot x_1 + \psi(y_2) + d \tag{1.27}$$

$$\begin{aligned} \dot{x}_2 &= F(x_1) \cdot x_2 + G(x_1) \cdot u \\ y_2 &= C_2 \cdot x_2 \end{aligned} \tag{1.28}$$

We will perform a first comparison with the feedback linearization general form to check if it can be applied [9-19].

$$\begin{aligned} \dot{x} &= f(x) + g(x) \cdot u \\ y &= h(x) \end{aligned} \tag{1.29}$$

The original system is indeed a special case of the one in equation (1.29) especially when undisturbed (**d** = **0**). Indeed the dynamic equation can be rewritten as follows:

$$\begin{aligned} \dot{\mathbf{x}} &= \underbrace{\begin{bmatrix} \mathbf{A}_{LP} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}(\mathbf{x}_1) \end{bmatrix} \mathbf{x} + \begin{bmatrix} \boldsymbol{\psi}(\mathbf{C}_2\mathbf{x}_2) \\ \mathbf{0} \end{bmatrix}}_{\mathbf{f}(\mathbf{x})} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{G}(\mathbf{x}_1) \end{bmatrix}}_{\mathbf{g}(\mathbf{x})} \mathbf{u}; \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \\ \mathbf{y} &\equiv \mathbf{y}_2 = \mathbf{C}_2 \cdot \mathbf{x}_2 \equiv h(\mathbf{x}) \end{aligned} \tag{1.30}$$

Also, note that the equivalent LP system introduced earlier is a special case of the one in equation (1.29) especially when undisturbed (d = 0). Indeed the dynamic equation can be rewritten as follows:

$$\begin{aligned} \dot{\mathbf{x}} &= \underbrace{\begin{bmatrix} \mathbf{A}_{LP} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}(\mathbf{x}_1) - j\omega_c\mathbf{I} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \boldsymbol{\psi}_{LP}(\mathbf{C}_2\tilde{\mathbf{x}}_2) \\ \mathbf{0} \end{bmatrix}}_{\mathbf{f}(\mathbf{x})} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{G}(\mathbf{x}_1) \end{bmatrix}}_{\mathbf{g}(\mathbf{x})} \tilde{\mathbf{u}}; \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \tilde{\mathbf{x}}_2 \end{bmatrix} \\ \mathbf{y} &\equiv \tilde{\mathbf{y}}_2 = \mathbf{C}_2 \cdot \tilde{\mathbf{x}}_2 \equiv h(\mathbf{x}) \end{aligned} \tag{1.31}$$

So the general procedure presented in [9-19] could be, at least in principle, attempted. However, an alternative approach will be pursued in the sequel. As can be seen the band-pass (BP) subsystem alone in equation (1.28) is clearly a special case of the affine nonlinear system structure in equation (1.29). However, the nonlinear cross-band coupling cannot occur unless the Taylor expansion of ψ hereafter does not contain even power terms; eventually, odd power terms play no role and are filtered out.

$$\boldsymbol{\psi}(\mathbf{y}_2) = \boldsymbol{\psi}(\mathbf{y}_2 = \mathbf{0}) + \left[\sum_{k=1}^{\infty} \boldsymbol{\psi}_i^{(k)} \cdot \mathbf{y}_2^{\otimes k} \right], 1 \le i \le n_1 \tag{1.32}$$

This is due to: (a) that transfer function matrix $H_2(s)$ as follows is BP around a sufficiently high carrier frequency ω_c:

$$\cdot \left(s\mathbf{I} - \mathbf{F}_0\right)^{-1} \cdot \boldsymbol{\Gamma} \tag{1.33}$$

Also, due to: (b) that transfer function matrix $H_1(s)$ as follows is LP:

$$\mathbf{H}_1(s) = \left(s\mathbf{I} - \mathbf{A}_{LP}\right)^{-1} \tag{1.34}$$

But linearization in any sense, e.g. exact input-state or input-output linearization via feedback or "traditional" approximate linearization around an (equilibrium) point in the state space, would essentially mean that the first power, which is odd would become the predominant feature of the intrinsic system structure. So in effect, if the LP subsystem is linearizable, too, then cross-band coupling with the BP subsystem through the even powers in the expansion of equation (1.32) for ψ might not be feasible. This is why the alternative approach is pursued here. For example, appropriate transformations as specified e.g. in [9-19] for input-state or input-output linearization may not exist or, at least, be meaningful to derive.

Initial structural considerations

We can now look into a simplistic and heuristic approach to BP subsystem exact feedback linearization and see how it fails but also what useful features can be adopted. For simplicity consider a single-input, two-dimensional case, aka:

$$\mathbf{x}_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \mathbf{F}(\mathbf{x}_1) = \begin{bmatrix} F_{11}(\mathbf{x}_1) & F_{12}(\mathbf{x}_1) \\ F_{21}(\mathbf{x}_1) & F_{22}(\mathbf{x}_1) \end{bmatrix}, \mathbf{G}(\mathbf{x}_1) = \begin{bmatrix} 0 \\ g(\mathbf{x}_1) \end{bmatrix}, g(\mathbf{x}_1) \ne 0 \tag{1.35}$$

Then, one might be tempted to try the following state feedback control toward exact linearization of the system:

$$u = \frac{1}{g(\mathbf{x}_1)}\left(v - [K_1 \quad K_2]\mathbf{F}(\mathbf{x}_1)\mathbf{x}_2\right) = \frac{v - [K_1 F_{11}(\mathbf{x}_1) + K_2 F_{21}(\mathbf{x}_1) \quad K_1 F_{12}(\mathbf{x}_1) + K_2 F_{22}(\mathbf{x}_1)]\mathbf{x}_2}{g(\mathbf{x}_1)} \tag{1.36}$$

In the above, K_1 and K_2 are gains to be determined. However, by substitution in the system's dynamical equation the following is obtained.

$$\dot{\mathbf{x}}_2 = \mathbf{F}(\mathbf{x}_1)\cdot\mathbf{x}_2 + \mathbf{G}(\mathbf{x}_1)\cdot\frac{v - [K_1 \quad K_2]\mathbf{F}(\mathbf{x}_1)\mathbf{x}_2}{g(\mathbf{x}_1)}$$
$$\Updownarrow \quad (1.37)$$
$$\left\{\begin{array}{c} \dot{x}_1 = x_1 F_{11}(\mathbf{x}_1) + x_2 F_{12}(\mathbf{x}_1) \\ \dot{x}_2 = x_1 F_{21}(\mathbf{x}_1) + x_2 F_{22}(\mathbf{x}_1) + v - (K_1 F_{11} + K_2 F_{21})x_1 - (K_1 F_{12} + K_2 F_{22})x_2 \end{array}\right\} \tag{1.37}$$

Clearly, setting $K_1 = 0$ and $K_2 = 1$ allows to linearize successfully the second equation of the above for x_2; however, the first equation for x_1 does not include either K_1 or K_2. In effect, it cannot be processed further.

Despite this failure, some elements of this straightforward analysis can be used further. Specifically, assume that F is of the following form.

$$\mathbf{F}(\mathbf{x}_1) = \begin{bmatrix} \zeta_1 & \zeta_2 \\ F_{21}(\mathbf{x}_1) & F_{22}(\mathbf{x}_1) \end{bmatrix}; \zeta_1, \zeta_2 \text{ real constants} \tag{1.38}$$

So effectively F is partitioned to a real and constant upper block (e.g. ζ_1, ζ_2 can be so that this block corresponds to the Brunovsky canonical form) and a nonlinear lower block depending exclusively on state vector x_1. In this case, after applying the feedback control law of equation (1.36) with the K's open, we see that the scalar equations for x_1 and x_2 become:

$$\begin{array}{l} \dot{x}_1 = \zeta_1 x_1 + \zeta_2 x_2 \\ \dot{x}_2 = (F_{21} - K_1 F_{11} - K_2 F_{21})x_1 + (F_{22} - K_1 F_{12} - K_2 F_{22})x_2 + v \end{array} \tag{1.39}$$

So now by setting $K_1 = 0$ and $K_2 = 1$ the equations become linear with respect to the synthetic input v as follows.

$$\dot{x}_1 = \zeta_1 x_1 + \zeta_2 x_2, \dot{x}_2 = v \tag{1.40}$$

A final point is now mentioned concerning the output equation of the BP subsystem.

$$\mathbf{y}_2 = \mathbf{C}_2 \cdot \mathbf{x}_2 \tag{1.41}$$

Its linear form may, as seen in the following section, allows dealing with input-state and input-output linearization in a unified fashion.

A sufficient band-pass subsystem structure

We now take advantage of the point made in the end of the previous section after generalizing and properly setting it up in the generic exact linearization framework introduced previously. Specifically, we first require setting the BP subsystem state-space equations into an appropriately modified controllable companion canonical form by taking into advantage the fact that matrix functions F and G of the dynamical equation do not depend on x_2 but only on the state vector of the LP subsystem x_1. This form will prove valuable to solve the partial linearization problem.

$$\dot{\mathbf{x}}_2 = \mathbf{F}(\mathbf{x}_1)\cdot\mathbf{x}_2 + \mathbf{G}(\mathbf{x}_1)\cdot\mathbf{u}$$
$$\mathbf{F}(\mathbf{x}_1) = \begin{bmatrix} \mathbf{F}_2 \\ \hline \mathbf{F}_1(\mathbf{x}_1) \end{bmatrix}, \mathbf{G}(\mathbf{x}_1) = \begin{bmatrix} \mathbf{0}_{\rho_2 \times m_2} \\ \hline \mathbf{G}_1(\mathbf{x}_1) \end{bmatrix}, \rho_2 = n_2 - m_2 \geq 0 \tag{1.42}$$

In the above:

$\rho_2 = n_2 - m_2$ i.e. the excess in number of states vs inputs of the BP subsystem

F_2 is a $\rho_2 \times n_2$ constant matrix

F_1 is a $m_2 \times n_2$ matrix function of LP subsystem state vector x_1

0 is the zero matrix or vector of specified dimensions

G_1 is a $m_2 \times m_2$ matrix function of LP subsystem state vector x_1; G_1 must be invertible in at least a domain $D_1 \subset \mathbb{R}^\kappa$ where $\kappa = n_1$ is the dimension of the LP subsystem state vector x_1.

Notice the similarity of the structure considered to the controllable companion canonical form introduced previously. Some further worth-mentioning points include first and foremost that the system structure considered here is, in general, multivariable. Moreover, constant matrix F_2 is actually a generalization of constant matrix A_{0I} in the Brunovsky canonical form [9-19]. Integer parameter "rho-two" (ρ_2) is an alternative to relative degree introduced in [9-19]; it is conveniently applicable to the affine subsystem of the overall nonlinear system with many inputs and outputs we are considering. Matrix block F_1 is a modified structural block holding the place of row matrix <u>a</u> in the Brunovsky canonical form [9-19]. Finally as can be seen, matrix G obtains a form similar to that of the product ($b_I\ \beta^{-1}$) arising e.g. in input-state linearization in [9-19]; more specifically, square block

$G_1(x_1)$ takes the place of $\beta^{-1}(z)$ in the general SISO exact input-state linearization framework.

Then, we demonstrate that if the following control input is applied to the system, the BP subsystem becomes linear in its state and with respect to the synthetic input vector v of dimension m_2, i.e. the same as u.

$$\mathbf{u} = \mathbf{G}_1^{-1}(\mathbf{x}_1)\cdot(\mathbf{v} - \mathbf{F}_1(\mathbf{x}_1)\cdot\mathbf{x}_2) \tag{1.43}$$

Indeed:

$$\begin{aligned}\dot{\mathbf{x}}_2 &= \mathbf{F}(\mathbf{x}_1)\cdot\mathbf{x}_2 + \mathbf{G}(\mathbf{x}_1)\cdot\mathbf{u} = \\ &= \left[\frac{\mathbf{F}_2}{\mathbf{F}_1(\mathbf{x}_1)}\right]\cdot\mathbf{x}_2 + \left[\frac{\mathbf{0}_{(n_2-m_2)\times m_2}}{\mathbf{G}_1(\mathbf{x}_1)}\right]\cdot\mathbf{G}_1^{-1}(\mathbf{x}_1)\cdot(\mathbf{v}-\mathbf{F}_1(\mathbf{x}_1)\cdot\mathbf{x}_2) = \\ &= \begin{bmatrix}\mathbf{F}_2\cdot\mathbf{x}_2 \\ \mathbf{F}_1(\mathbf{x}_1)\cdot\mathbf{x}_2\end{bmatrix} + \begin{bmatrix}\bar{\mathbf{0}}_{(n_2-m_2)} \\ \mathbf{v}-\mathbf{F}_1(\mathbf{x}_1)\cdot\mathbf{x}_2\end{bmatrix} = \begin{bmatrix}\mathbf{F}_2\cdot\mathbf{x}_2 \\ \mathbf{v}\end{bmatrix}\end{aligned}$$

So in effect, the BP subsystem equations after exact feedback linearization is applied according to the procedure presented become as follows.

$$\begin{aligned}\dot{\mathbf{x}}_2 &= \left[\frac{\mathbf{F}_2}{\mathbf{0}_{m_2\times n_2}}\right]\cdot\mathbf{x}_2 + \left[\frac{\mathbf{0}_{(n_2-m_2)\times m_2}}{\mathbf{I}_{m_2\times m_2}}\right]\cdot\mathbf{v} = \Theta\mathbf{x}_2 + \mathrm{B}_{0\mathrm{I}}\mathbf{v} \\ \mathbf{y}_2 &= \mathbf{C}_2\cdot\mathbf{x}_2\end{aligned} \tag{1.44}$$

Note that the system is automatically linearized in the input-output sense too since the output equation in the above is linear in the first place by assumption.

Exact linearization of the low-pass equivalent

We now proceed and extend the exact linearization method established in the previous section for the BP subsystem to its LP equivalent. We will consider the pre-envelope and the complex envelope for the BP subsystem's state and input vector; when we first introduced these concepts we saw that by using the pre-envelope, the complex envelope can be defined for a BP signal vector with carrier frequency $\omega_c = 2\pi f_c$.

$$\begin{aligned}\mathbf{x}_{2+}(t) &= \tilde{\mathbf{x}}_2(t)\exp(j\omega_c t) \\ \mathbf{u}_+(t) &= \tilde{\mathbf{u}}(t)\exp(j\omega_c t)\end{aligned} \tag{1.45}$$

As explained previously, the pre-envelope is a complex signal vector allowed to have nonzero spectrum only in non-negative frequencies. Furthermore, as we saw earlier a similar pattern can be applied to the output vector of the BP subsystem as follows.

$$\left.\begin{aligned}\mathbf{y}_{2+}(t) &= \tilde{\mathbf{y}}_2(t)\exp(j\omega_c t) \\ \mathbf{y}_2(t) &= \mathrm{Re}\{\mathbf{y}_{2+}(t)\}\end{aligned}\right\} \Rightarrow \tilde{\mathbf{y}}_2 = \mathbf{C}_2\cdot\tilde{\mathbf{x}}_2 \tag{1.46}$$

The last one of the above relationships in combination with the fact that signals u, x_2 and consequently y_2 are BP with carrier frequency $\omega_c = 2\pi f_c$ leads to the conclusion that their complex are LP signals. So for the (LP) complex envelopes the BP subsystem equations become the LP equivalent system of equations: (2.21)

$$\begin{aligned}\dot{\tilde{\mathbf{x}}}_2 &= \left[\mathbf{F}(\mathbf{x}_1) - j\omega_c\mathbf{I}\right]\cdot\tilde{\mathbf{x}}_2 + \mathbf{G}(\mathbf{x}_1)\cdot\tilde{\mathbf{u}} \\ \tilde{\mathbf{y}}_2 &= \mathbf{C}_2\cdot\tilde{\mathbf{x}}_2\end{aligned} \tag{1.47}$$

We then proceed to assume as in the previous section that matrices F and G have the following form.

$$\mathbf{F}(\mathbf{x}_1) = \left[\frac{\mathbf{F}_2}{\mathbf{F}_1(\mathbf{x}_1)}\right], \mathbf{G}(\mathbf{x}_1) = \left[\frac{\mathbf{0}_{(n_2-m_2)\times m_2}}{\mathbf{G}_1(\mathbf{x}_1)}\right] \tag{1.48}$$

Again:

$\rho_2 = n_2 - m_2$ i.e. the excess in number of states vs inputs of the BP subsystem

F_2 is a $\rho_2 \times n_2$ constant matrix

F_1 is a $m_2 \times n_2$ matrix function of LP subsystem state vector x_1

0 is the zero matrix or vector of specified dimensions

G_1 is a $m_2 \times m_2$ matrix function of LP subsystem state vector x_1; G_1 must be invertible in at least a domain $D_1 \subset \mathbb{R}^\kappa$ where $\kappa = n_1$ is the dimension of the LP subsystem state vector x_1.

We now evaluate the form of the system when the control input u is chosen as follows.

$$\left.\begin{aligned}\mathbf{u} &= \mathbf{G}_1^{-1}(\mathbf{x}_1)\cdot(\mathbf{v}-\mathbf{F}_1(\mathbf{x}_1)\cdot\mathbf{x}_2) \\ &\Updownarrow \\ \hat{\mathbf{u}} &= \mathbf{G}_1^{-1}(\mathbf{x}_1)\cdot(\hat{\mathbf{v}}-\mathbf{F}_1(\mathbf{x}_1)\cdot\hat{\mathbf{x}}_2)\end{aligned}\right\} \Rightarrow \mathbf{u}_+ = \mathbf{G}_1^{-1}(\mathbf{x}_1)\cdot(\mathbf{v}_+ - \mathbf{F}_1(\mathbf{x}_1)\cdot\mathbf{x}_{2+}) \tag{1.49}$$

Then, we demonstrate that by employing the control input above, the BP subsystem's LP equivalent becomes linear in its state's complex envelope vector and with respect to the synthetic input's complex envelope vector. As before, the synthetic input v has dimension m_2, i.e. the same as u. Furthermore for its Hilbert Transform, pre-envelope and complex envelope the following hold.

$$\hat{v}(t) \triangleq \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{t-t_1}v(t_1)dt_1 = \frac{1}{\pi t} * v(t) \Leftrightarrow \hat{v}(f) = -j\,sgn(f)v(f)$$

$$v_+(t) \triangleq v(t) + j\hat{v}(t) = \tilde{v}(t)\,exp(j\omega_c t) \Leftrightarrow \tilde{v}(t) \triangleq \left(v(t) + j\hat{v}(t)\right)exp(-j\omega_c t)$$

The complex envelope of synthetic input v may be decomposed to a real and an imaginary component. In telecommunications literature the real component, $v_C(t)$, is referred to as the in-phase (or I for short) component and the imaginary component, $v_S(t)$, is referred to as the quadrature (or Q for short) component. Clearly, the I and Q components of the complex envelope are mutually orthogonal and preserve the complete information content of the BP signal from which they are generated. Furthermore, as can be seen from the properties of Hilbert Transform and the complex envelope [21], the complex envelope is a generalization of the concept of amplitude modulation applied to the generalized imaginary exponential carrier signal $\exp(j\omega_c t)$.

$$\tilde{\mathbf{v}}(t) = \mathbf{v}_C(t) + j\mathbf{v}_S(t) \tag{1.50}$$

Finally, it is noted that the spectrum of the complex envelope of the synthetic input v is LP as expected. Indeed for the Fourier Transform of v the following can be derived.

$$\tilde{\mathbf{v}}(f) = \mathbf{v}_+(f - f_c) = 2\mathbf{v}(f - f_c) \tag{1.51}$$

We proceed now to use the complex envelope of the control input in equation (1.49) above.

$$\begin{gathered} \tilde{\mathbf{u}}e^{j\omega_c t} = \mathbf{G}_1^{-1}(\mathbf{x}_1)\cdot\left(\tilde{\mathbf{v}}e^{j\omega_c t} - \mathbf{F}_1(\mathbf{x}_1)\cdot\tilde{\mathbf{x}}_2 e^{j\omega_c t}\right) \\ \Updownarrow \\ \tilde{\mathbf{u}} = \mathbf{G}_1^{-1}(\mathbf{x}_1)\cdot\left(\tilde{\mathbf{v}} - \mathbf{F}_1(\mathbf{x}_1)\cdot\tilde{\mathbf{x}}_2\right) \end{gathered} \tag{1.52}$$

We now substitute the above in equation (1.47) and proceed as follows.

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}_2 &= \left[\mathbf{F}(\mathbf{x}_1) - j\omega_c\mathbf{I}\right]\cdot\tilde{\mathbf{x}}_2 + \mathbf{G}(\mathbf{x}_1)\cdot\tilde{\mathbf{u}} \\ &= \left(\left[\frac{\mathbf{F}_2}{\mathbf{F}_1(\mathbf{x}_1)}\right] - j\omega_c\mathbf{I}\right)\cdot\tilde{\mathbf{x}}_2 + \left[\frac{\mathbf{0}_{(n_2-m_2)\times m_2}}{\mathbf{G}_1(\mathbf{x}_1)}\right]\cdot\mathbf{G}_1^{-1}(\mathbf{x}_1)\cdot\left(\tilde{\mathbf{v}} - \mathbf{F}_1(\mathbf{x}_1)\cdot\tilde{\mathbf{x}}_2\right) = \\ &= \begin{bmatrix} \mathbf{F}_2\cdot\tilde{\mathbf{x}}_2 \\ \mathbf{F}_1(\mathbf{x}_1)\cdot\tilde{\mathbf{x}}_2 \end{bmatrix} - j\omega_c\tilde{\mathbf{x}}_2 + \begin{bmatrix} \vec{\mathbf{0}}_{(n_2-m_2)} \\ \tilde{\mathbf{v}} - \mathbf{F}_1(\mathbf{x}_1)\cdot\tilde{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{F}_2\cdot\tilde{\mathbf{x}}_2 \\ \tilde{\mathbf{v}} \end{bmatrix} - j\omega_c\tilde{\mathbf{x}}_2 \end{aligned}$$

In effect the following linearized LP equivalent is obtained.

$$\begin{gathered} \dot{\tilde{\mathbf{x}}}_2 = \left[\frac{\mathbf{F}_2}{\mathbf{0}_{m\times n}}\right]\cdot\tilde{\mathbf{x}}_2 - j\omega_c\tilde{\mathbf{x}}_2 + \left[\frac{\mathbf{0}_{(n-m)\times m}}{\mathbf{I}_{m\times m}}\right]\cdot\tilde{\mathbf{v}} = \left[\Theta - j\omega_c\mathbf{I}\right]\tilde{\mathbf{x}}_2 + \mathrm{B}_{0\mathrm{I}}\tilde{\mathbf{v}} \Rightarrow \\ \Downarrow \\ \begin{Bmatrix} \dot{\tilde{\mathbf{x}}}_2 = \Theta_{LP}\tilde{\mathbf{x}}_2 + \mathrm{B}_{0\mathrm{I}}\tilde{\mathbf{v}} \\ \tilde{\mathbf{y}}_2 = \mathbf{C}_2\cdot\tilde{\mathbf{x}}_2 \end{Bmatrix} \end{gathered} \tag{1.53}$$

The above means that if the original BP subsystem is amenable to exact feedback linearization so is its LP equivalent at least in the manner presented in the context developed in this work.

Electromechanical system applications

Electromechanical system governing equations

We return now to the system introduced earlier that exhibits nonlinear cross-band coupling and state modulation and demodulation. By using first principles for the system of figure 2, the following Lagrangian, L_L, is obtained for the non-dissipative and unforced case (R = 0, b = 0 and e = 0) [20-24]:

$$\mathrm{L}_L(q,\dot{q},y,\dot{y}) = \frac{1}{2}L(y)\dot{q}^2 + \frac{1}{2}m\dot{y}^2 - \frac{1}{2C}q^2 - \frac{1}{2}ky^2 \tag{1.54}$$

In the above q(t) denotes the capacitor's charge and, therefore, $\dot{q}(t) = i(t)$ is the circuit's current; $x(t), \dot{x}(t)$ are the payload displacement position and velocity, respectively; L is the electromagnet's inductance, R and C the circuit resistance and capacitance; m is the payload mass, b the damping coefficient and k the spring constant of the mechanical oscillator. Then, we introduce the following canonical coordinates [20-24]:

$$p_y = \frac{\partial \mathrm{L}_L}{\partial \dot{y}} = m\dot{y} \tag{1.55}$$

$$p_q = \frac{\partial \mathrm{L}_L}{\partial \dot{q}} = L(y)\cdot\dot{q} \tag{1.56}$$

In result, the Hamiltonian, H, is obtained as follows [20-24]:

$$\mathrm{H}(q,p_q,y,p_y) = \frac{1}{2}\cdot\frac{p_q^{\,2}}{L(y)} + \frac{1}{2}\cdot\frac{p_y^{\,2}}{m} + \frac{1}{2C}\cdot q^2 + \frac{1}{2}\cdot ky^2 \tag{1.57}$$

The Hamiltonian describes the non-dissipative system. For the actual system, including the damping terms and excitation (forcing) the following set of equations is derived [20-24]:

$$i \triangleq \dot{q} = \frac{p_q}{L(y)} \tag{1.58}$$

$$\dot{y} = \frac{p_y}{m} \tag{1.59}$$

$$p_q = -\frac{1}{C}\cdot q - p_q\cdot\frac{R}{L(y)} \tag{1.60}$$

$$\dot{p}_y = -ky - p_y\frac{b}{m} + \frac{1}{2}\cdot\frac{p_q^{\,2}}{L^2(y)} \tag{1.61}$$

In effect, one obtains the following second-order dynamical equations for the electrical and mechanical subsystems:

$$L_1\left[\dot{y}(t)\cdot\dot{q}(t)+y(t)\cdot\ddot{q}(t)\right]+L_0\ddot{q}(t)+R\dot{q}(t)+\frac{1}{C}q(t)=e(t) \quad (1.62)$$

$$m\ddot{y}(t)+b\dot{y}(t)+ky(t)=\frac{dL}{dy}\cdot\frac{i^2(t)}{2} \quad (1.63)$$

The above system consists of two coupled second-order oscillators. However, the coupling is nonlinear. Indeed, the right-hand magnetic force term of equation (1.63) and the first term of the left-hand side of equation (1.62) are clearly nonlinear.

Furthermore, if the inductance were not a function of the payload displacement, then coupling would not take place. A common dependence of the inductance on payload displacement that can be justified by electromechanical theory and analysis of magnetic circuits is the following [20-24]:

$$L(y)=L_0+L_1y, 0\le y\le y_{em} \quad (1.64)$$

The equation above is valid only a limited finite range of mass displacement y. It comes only as an approximation to the commonly encountered sigmoid dependence [20-24], like e.g. that depicted by the logistic sigmoid, of inductance to the position of a metallic mass like the one considered here. In specific, a relationship for L(y) valid over the entire real value range for y might look like the following.

$$L(y)=L_\infty+\frac{(\chi-1)L_\infty}{1+\exp\left[\frac{2y_{em}L_1}{(\chi-1)L_\infty}\left(1-2\frac{y}{y_{em}}\right)\right]}, \chi>1 \quad (1.65)$$

Based on the above one obtains the following facts.

$$\begin{aligned}
&L(y\to-\infty)=L_\infty, L(y\to+\infty)=\chi L_\infty,\\
&L\left(\frac{y_{em}}{2}\right)=\frac{\chi+1}{2}L_\infty\\
&L(0)=L_\infty+\frac{(\chi-1)L_\infty}{1+\exp\left(\frac{2y_{em}L_1}{(\chi-1)L_\infty}\right)}\\
&L(y_{em})=L_\infty+\frac{(\chi-1)L_\infty}{1+\exp\left(-\frac{2y_{em}L_1}{(\chi-1)L_\infty}\right)}
\end{aligned} \quad (1.66)$$

By applying Taylor's expansion to equation (1.65) around point and keeping only the first order term one obtains the following.

$$L(y)=\frac{\chi+1}{2}L_\infty+L_1\left(y-\frac{y_{em}}{2}\right)=\underbrace{\frac{(\chi+1)L_\infty-L_1y_{em}}{2}}_{L_0}+L_1y \quad (1.67)$$

Without any loss of generality assume that the electromagnet in figure 2 is placed at a position on the y-axis equal to the characteristic length y_{em}. The characteristic length can be calculated as the distance between the position where the mass is inductively decoupled from the electromagnet and the position where it has the maximum effect on the inductance.

$$y_{em}=\frac{L(+\infty)-L(-\infty)}{L_1}=\frac{\chi-1}{L_1}L_\infty \quad (1.68)$$

Using the above, the following observations should be noted.

$$\begin{aligned}
&L(y_{em})\approx L(+\infty), L(0)\approx L(-\infty), L_0=L_\infty\\
&L(y)\simeq\begin{cases}L_\infty, y<0\\ L_\infty+L_1y, 0\le y\le y_{em}\\ L_\infty+L_1y_{em}, y>y_{em}\end{cases}
\end{aligned} \quad (1.69)$$

So provided that the mass displacement is constrained within the interval indicated in equation (1.64), the central branch of the equation above can be used. In the system that will be considered in the remaining of this text, hard stoppers will be employed to ensure that the mass displacement remains within the permissible range. Furthermore, for the sake of simplicity and without loss of generality, the electromagnet will be placed at position y_{em} and the natural length of the mechanism's spring will correspond to y = 0, while y is assumed to increase from the spring's natural length toward the electromagnet. By using equation (1.64) in equations (1.62) and (1.63) one obtains:

$$L_1\left[\dot{y}(t)\cdot\dot{q}(t)+y(t)\cdot\ddot{q}(t)\right]+L_0\ddot{q}(t)+R\dot{q}(t)+\frac{1}{C}q(t)=e(t) \quad (1.70)$$

$$m\ddot{y}(t)+b\dot{y}(t)+ky(t)=\frac{L_1}{2}i^2(t), 0\le y\le y_{em} \quad (1.71)$$

The nonlinear coupling between the two oscillators is quantified clearly in (1.65) and (1.66). Furthermore, as can be seen decoupling occurs if $L_1 = 0$.

Formulation of system dynamics in state space

As seen in the previous section, for the analysis of the system of coupled electromechanical oscillators shown in figure 2, a few basic principles were employed to obtain the following set of nonlinear, second order, ordinary differential equations for payload displacement y and capacitor charge q:

$$m\ddot{y}+b\dot{y}+ky=\frac{L_1}{2}i^2+d \quad (1.72)$$

$$\left[L_0+L_1y\right]\ddot{q}+\left[R+L_1\dot{y}\right]\dot{q}+\frac{1}{C}q=e \quad (1.73)$$

In the above, the force disturbance signal d has been superimposed to the electromagnet's force in the LP mechanical subsystem. As it is common practice for electromechanical systems the following state vector may be used:

$$\mathbf{x} = \begin{bmatrix} y & \dot{y} & q & \dot{q} \end{bmatrix}^T \tag{1.74}$$

This is partitioned as follows for establishing the correspondence with the LP-BP formulation presented earlier:

$$\mathbf{x}_1 = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}, \mathbf{d} = \begin{bmatrix} 0 \\ \dfrac{d}{m} \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}, \mathbf{u} = e, \mathbf{y}_2 = i \tag{1.75}$$

By using the above, the following matrices are obtained for the BP-LP state space decomposition when applied to the case of the coupled electromechanical oscillators at hand:

$$\begin{gathered} \mathbf{A}_{LP} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix}, \boldsymbol{\psi}(\mathbf{y}_2) = \begin{bmatrix} 0 \\ \dfrac{L_1}{2m} i^2 \end{bmatrix}, \\ \mathbf{F}(\mathbf{x}_1) = \begin{bmatrix} 0 & 1 \\ -\dfrac{1}{(L_0 + L_1 y)C} & -\dfrac{R + L_1 \dot{y}}{L_0 + L_1 y} \end{bmatrix}, \\ \mathbf{G}(\mathbf{x}_1) = \begin{bmatrix} 0 \\ \dfrac{1}{L_0 + L_1 y} \end{bmatrix}, \mathbf{C}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \end{gathered} \tag{1.76}$$

As can be seen in this case, $\psi(y_2)$ is already in the required expansion form specified previously; therefore, no further treatment is needed. Otherwise, transfer function matrix $H_1(s) = (sI - A)^{-1}$ is given by the following: transfer function matrix $H_1(s) = (sI - A)^{-1}$ is given by the following:

$$\mathbf{H}_1(s) = \frac{1}{ms^2 + bs + k} \begin{bmatrix} (ms + b) & m \\ k & ms \end{bmatrix} \tag{1.77}$$

The poles of the above are the roots of (characteristic) polynomial $P_1(s) = ms^2 + bs + k$. It can also be seen that:

$$\mathbf{H}_1(s = j0) = \begin{bmatrix} \dfrac{b}{k} & \dfrac{m}{k} \\ 1 & 0 \end{bmatrix} \tag{1.78}$$

For frequency s = jω going to infinity, one can straightforwardly verify that all scalar transfer functions in the entries of transfer function matrix $H_1(s)$ vanish. Therefore, the LP requirement for $H_1(s)$ clearly checks out. In a more general case, however, one should employ Singular Value Decomposition as outlined earlier in order to establish whether $H_1(s)$ is LP or not.

We now need to proceed to establish whether transfer function matrix $H_2(s) = C_2\,(sI - F_0)^{-1}\,\Gamma$ is BP or not. In this respect, we need to calculate matrices F_0 and Γ. This can be achieved by employing their Taylor expansion as outlined earlier. For the matrix Taylor expansions, though, the following scalar one is useful:

$$\frac{1}{L_0 + L_1 y} = \frac{1}{L_0} - \frac{L_1}{L_0^{\,2}} y + \frac{L_1^{\,2}}{L_0^{\,3}} y^2 - \frac{L_1^{\,3}}{L_0^{\,4}} y^3 + \cdots \tag{1.79}$$

Furthermore, the above yields:

$$\frac{1}{L_0 + L_1 y} = \frac{1}{L_0\left(1 + \dfrac{L_1}{L_0} y\right)} = \frac{1}{L_0}\left(1 - \frac{L_1}{L_0} y + \frac{L_1^{\,2}}{L_0^{\,2}} y^2 - \frac{L_1^{\,3}}{L_0^{\,3}} y^3 + \cdots\right) \tag{1.80}$$

For micro- (or even nano-) electromechanical systems (MEMS or NEMS) [6,10] as well as in other applied mechatronics [4,5] it is reasonable to assume that $L_1 y \ll L_0$. With this in mind, the following first-order approximation will be considered in the above:

$$\frac{1}{L_0 + L_1 y} \cong \frac{1}{L_0}\left(1 - \frac{L_1}{L_0} y\right) = \frac{1}{L_0} - \frac{L_1}{L_0^2} y \tag{1.81}$$

Finally, by employing the above one obtains the following for multivariable matrix functions F and G.

$$\mathbf{F}(\mathbf{x}_1) = \underbrace{\begin{bmatrix} 0 & 1 \\ -\dfrac{1}{CL_0} & -\dfrac{R}{L_0} \end{bmatrix}}_{\mathbf{F}_0} + \underbrace{\begin{bmatrix} 0 & 0 \\ \dfrac{L_1}{CL_0^{\,2}} y & \begin{bmatrix} \dfrac{RL_1}{L_0^{\,2}} & -\dfrac{L_1}{L_0} \end{bmatrix} \cdot \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \end{bmatrix}}_{\mathbf{F}_1(\mathbf{x}_1^{\otimes 1})} \tag{1.82}$$

$$\mathbf{G}(\mathbf{x}_1) = \underbrace{\begin{bmatrix} 0 \\ \dfrac{1}{L_0} \end{bmatrix}}_{\mathbf{G}_0} + \underbrace{\begin{bmatrix} 0 \\ -\dfrac{L_1}{L_0^{\,2}} y \end{bmatrix}}_{\mathbf{G}_1(\mathbf{x}_1^{\otimes 1})} \tag{1.83}$$

By using the above one obtains the following for matrix Γ.

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{G}_0 & \tilde{\mathbf{A}}_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \dfrac{1}{L_0} & 1 & 1 \end{bmatrix} \tag{1.84}$$

Notice that the above form for Γ as well the one for Γ_0 does not change even if the approximation in equation (1.81) is dropped.

So in result the following is obtained for transfer function matrix $H_2(s)$.

$$\mathbf{H}_2(s) = \mathbf{C}_2 \cdot (s\mathbf{I} - \mathbf{F}_0)^{-1} \cdot \boldsymbol{\Gamma} = \frac{Cs}{L_0 Cs^2 + RCs + 1}\begin{bmatrix} 1 & L_0 & L_0 \end{bmatrix} \quad (1.85)$$

The poles of the above are the roots of (characteristic) polynomial $P_2(s) = L_0Cs^2 + RCs + 1$. It can also be seen that:

$$H_2(s = j0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad (1.86)$$

Also, as frequency s = jω is going to infinity, one can straightforwardly verify that all scalar transfer functions in the entries of transfer function matrix $H_2(s)$ vanish. Actually, only in a vicinity of frequency ω_E, defined below, the elements of $H_2(s)$ assume magnitude-wise non-negligible values. Therefore, the BP requirement with carrier frequency $\omega_c = \omega_E$ for $H_2(s)$ is therefore established.

$$\omega_c = \omega_E = \sqrt{\frac{1}{CL_0}} \quad (1.87)$$

The values of the entries in $H_2(s)$ when $\omega = \omega_c = \omega_E$ are given by the following.

$$\mathbf{H}_2(j\omega_E) = \frac{1}{R}\begin{bmatrix} 1 & L_0 & L_0 \end{bmatrix} \quad (1.88)$$

In summary, the LP equivalent of the system consisting of the two coupled electromechanical oscillators at hand is given by the following.

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \cdot \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{L_1}{2m} \cdot \frac{|\tilde{i}|^2}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{d}{m} \end{bmatrix} \quad (1.89)$$

$$\frac{d}{dt}\begin{bmatrix} \tilde{q} \\ \tilde{i} \end{bmatrix} = \begin{bmatrix} -j\omega_E & 1 \\ -\frac{1}{(L_0 + L_1 y)C} & -\frac{R + L_1\dot{y}}{L_0 + L_1 y} - j\omega_E \end{bmatrix} \cdot \begin{bmatrix} \tilde{q} \\ \tilde{i} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_0 + L_1 y} \end{bmatrix} \tilde{e} \quad (1.90)$$

$$\tilde{i} = \tilde{\dot{q}} = \frac{d\tilde{q}}{dt} + j\omega_E \tilde{q} \Rightarrow \frac{d\tilde{q}}{dt} = -j\omega_E \tilde{q} + \tilde{i} \quad (1.91)$$

The above is the full LP equivalent system, meaning that matrices F and G are used in full and not by some approximation. If the first-order approximations in equations (1.82) and (1.83) are employed instead in the dynamics of the LP equivalent in equation (1.90) perturbation analysis may be carried out [1,20-21]. Perturbation analysis clearly reveals that by using the LP equivalent system the solutions obtained are identical for the electromechanical coupled oscillators at hand [1,20-21]. However, in the sequel we intend to apply the modified exact feedback linearization approach introduced earlier for nonlinear systems with state modulation. We will then run numerical simulations to confirm the approach.

Exact feedback linearization of the electromechanical system

We now proceed and apply the feedback linearization approach to the BP subsystem of the coupled electromechanical oscillators, which is no other than the electrical part of the system. We recall here that:

$$\mathbf{A}_{LP} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \boldsymbol{\psi}(\mathbf{y}_2) = \begin{bmatrix} 0 \\ \frac{L_1}{2m} i^2 \end{bmatrix},$$
$$\mathbf{F}(\mathbf{x}_1) = \begin{bmatrix} 0 & 1 \\ -\frac{1}{(L_0 + L_1 y)C} & -\frac{R + L_1\dot{y}}{L_0 + L_1 y} \end{bmatrix}, \quad (1.92)$$
$$\mathbf{G}(\mathbf{x}_1) = \begin{bmatrix} 0 \\ \frac{1}{L_0 + L_1 y} \end{bmatrix}, \mathbf{C}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Then, the following can be derived concerning the required structure toward BP linearization.

$$n_2 = 2, m_2 = 1, \rho_2 = n_2 - m_2 = 1$$
$$\mathbf{F}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \mathbf{F}_1(\mathbf{x}_1) = \begin{bmatrix} -\frac{1}{(L_0 + L_1 y)C} & -\frac{R + L_1\dot{y}}{L_0 + L_1 y} \end{bmatrix}, \mathbf{G}_1(\mathbf{x}_1) = \frac{1}{L_0 + L_1 y} \quad (1.93)$$

In effect, one obtains that:

$$\mathbf{F}(\mathbf{x}_1) = \begin{bmatrix} 0 & 1 \\ \hline -\frac{1}{(L_0 + L_1 y)C} & -\frac{R + L_1\dot{y}}{L_0 + L_1 y} \end{bmatrix}, \mathbf{G}(\mathbf{x}_1) = \begin{bmatrix} 0 \\ \hline \frac{1}{L_0 + L_1 y} \end{bmatrix} \quad (1.94)$$

In effect the following control input allows the exact linearization with respect to both the states and the output of the BP subsystem.

$$\mathbf{u} = \mathbf{G}_1^{-1}(\mathbf{x}_1) \cdot (\mathbf{v} - \mathbf{F}_1(\mathbf{x}_1) \cdot \mathbf{x}_2)$$
$$\Downarrow$$
$$e = (L_0 + L_1 y)\left(\mathrm{q_{tt}} + \frac{q}{(L_0 + L_1 y)C} + \frac{(R + L_1\dot{y})i}{L_0 + L_1 y} \right) \quad (1.95)$$
$$\Downarrow$$
$$e = (L_0 + L_1 y)\mathrm{q_{tt}} + C^{-1}q + (R + L_1\dot{y})i$$

In the above, q_{tt} is the synthetic input with respect to which the BP subsystem is expected to be linear. Indeed, by using equation (1.95) the BP subsystem dynamic equation become as follows.

$$L_1(\dot{y}i + y\ddot{q}) + L_0\ddot{q} + Ri + \frac{1}{C}q = (L_0 + L_1 y)\mathrm{q_{tt}} + C^{-1}q + (R + L_1\dot{y})i$$
$$\Downarrow$$
$$(L_0 + L_1 y)\ddot{q} = (L_0 + L_1 y)\mathrm{q_{tt}} \quad (1.96)$$
$$\Downarrow$$
$$\ddot{q} = \mathrm{q_{tt}}$$

In the above, synthetic input q_{tt} has units of electric current rate i.e. A/s (or equivalently C/s^2) in the International System of Units (SI). The linear state equations of the BP system when the above linearizing control voltage, e, is used become just a couple of integrators connected in tandem, as follows.

$$\begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} q_{tt} \qquad (1.97)$$
$$i = \dot{q} \Rightarrow \frac{di}{dt} = \ddot{q} = q_{tt}$$

So now the current through the circuit that controls the voltage applied on the payload of the mechanical LP subsystem is directly controlled by the synthetic input q_{tt} in a straightforward linear manner. Furthermore, notice that the linearized system is in the Brunovsky canonical form [9-19] with two integrators.

For the LP equivalent, a similar pattern arises with of course the complex envelope of the synthetic input, i.e. the complex envelope of the electric current rate, replacing q_{tt} in the equations. In specific, for the LP equivalent of the BP subsystem the control input to be used is as follows.

$$\begin{gathered} \tilde{\mathbf{u}} = \mathbf{G}_1^{-1}(\mathbf{x}_1)\cdot\left(\tilde{\mathbf{v}} - \mathbf{F}_1(\mathbf{x}_1)\cdot\tilde{\mathbf{x}}_2\right) \\ \Downarrow \\ \tilde{e} = (L_0 + L_1 y)\left(\tilde{q}_{tt} + \frac{\tilde{q}}{(L_0 + L_1 y)C} + \frac{(R + L_1\dot{y})\tilde{i}}{L_0 + L_1 y}\right) \\ \Downarrow \\ \tilde{e} = (L_0 + L_1 y)\tilde{q}_{tt} + C^{-1}\tilde{q} + (R + L_1\dot{y})\tilde{i} \end{gathered} \qquad (1.98)$$

So now the state equations of the BP subsystem become as follows.

$$\frac{d}{dt}\begin{bmatrix} \tilde{q} \\ \tilde{i} \end{bmatrix} = \begin{bmatrix} -j\omega_E & 1 \\ -\frac{1}{(L_0 + L_1 y)C} & -\frac{R + L_1\dot{y}}{L_0 + L_1 y} - j\omega_E \end{bmatrix} \begin{bmatrix} \tilde{q} \\ \tilde{i} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L_0 + L_1 y} \end{bmatrix} \left((L_0 + L_1 y)\tilde{q}_{tt} + C^{-1}\tilde{q} + (R + L_1\dot{y})\tilde{i}\right) \qquad (1.99)$$

In effect one obtains the following.

$$\begin{bmatrix} \frac{d\tilde{q}}{dt} \\ \frac{d\tilde{i}}{dt} \end{bmatrix} = \begin{bmatrix} -j\omega_E\tilde{q} + \tilde{i} \\ -\frac{\tilde{q}}{(L_0 + L_1 y)C} - \left(\frac{R + L_1\dot{y}}{L_0 + L_1 y} + j\omega_E\right)\tilde{i} \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{q}_{tt} + \frac{\tilde{q}}{(L_0 + L_1 y)C} + \frac{(R + L_1\dot{y})\tilde{i}}{L_0 + L_1 y} \end{bmatrix} \qquad (1.100)$$

The above can be further decomposed to the following scalar linear, first-order ordinary differential equations:

$$\left.\begin{aligned} \frac{d\tilde{q}}{dt} + j\omega_E\tilde{q} &= \tilde{i} \\ \frac{d\tilde{i}}{dt} + j\omega_E\tilde{i} &= \tilde{q}_{tt} \end{aligned}\right\} \Rightarrow \frac{d}{dt}\begin{bmatrix} \tilde{q} \\ \tilde{i} \end{bmatrix} = \begin{bmatrix} -j\omega_E & 1 \\ 0 & -j\omega_E \end{bmatrix} \begin{bmatrix} \tilde{q} \\ \tilde{i} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{q}_{tt} \qquad (1.101)$$

The equations above for the linearized LP equivalent are not in the Brunovsky canonical form, like for the original system when exact feedback linearization is employed, but rather in the Jordan canonical form. Both the Brunovsky form for the BP subsystem and the Jordan form for its LP equivalent reflect the original structure of the electrical subsystem as it arises in the coupled oscillators.

Equations (1.101) can also be derived independently and directly from equations (1.97) that depict the dynamics of the original coupled system after the modified exact feedback linearization procedure has been carried out. Indeed:

$$\left.\begin{aligned} \dot{q} &= \frac{d}{dt}\mathrm{Re}\left(\tilde{q}e^{j\omega_E t}\right) = \frac{d}{dt}\mathrm{Re}\left(\tilde{q}^* e^{-j\omega_E t}\right) = \frac{d}{dt}\left(\frac{\tilde{q}e^{j\omega_E t} + \tilde{q}^* e^{-j\omega_E t}}{2}\right) = \\ &= \left(\frac{d\tilde{q}}{dt} + j\omega_E\tilde{q}\right)\frac{e^{j\omega_E t}}{2} + \left(\frac{d\tilde{q}^*}{dt} - j\omega_E\tilde{q}^*\right)\frac{e^{-j\omega_E t}}{2} \\ \dot{q} &= \frac{\tilde{\dot{q}}e^{j\omega_E t} + \tilde{\dot{q}}^* e^{-j\omega_E t}}{2}, \tilde{i} = \tilde{\dot{q}} \Rightarrow \dot{q} = \frac{\tilde{i}e^{j\omega_E t} + \tilde{i}^* e^{-j\omega_E t}}{2} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{aligned} \frac{d\tilde{q}}{dt} + j\omega_E\tilde{q} &= \tilde{\dot{q}} = \tilde{i} \\ \frac{d\tilde{q}^*}{dt} - j\omega_E\tilde{q}^* &= \tilde{\dot{q}}^* = \tilde{i}^* \end{aligned}\right\} \Rightarrow \frac{d\tilde{q}}{dt} + j\omega_E\tilde{q} = \tilde{i}$$

$$\left.\begin{aligned} \frac{di}{dt} &= \ddot{q} = q_{tt} = \mathrm{Re}\left(\tilde{q}_{tt}e^{j\omega_E t}\right) = \frac{\tilde{q}_{tt}e^{j\omega_E t} + \tilde{q}_{tt}^* e^{-j\omega_E t}}{2} \\ \frac{di}{dt} &= \frac{d}{dt}\mathrm{Re}\left(\tilde{i}e^{j\omega_E t}\right) = \frac{d}{dt}\left(\frac{\tilde{i}e^{j\omega_E t} + \tilde{i}^* e^{-j\omega_E t}}{2}\right) \\ &= \left(\frac{d\tilde{i}}{dt} + j\omega_E\tilde{i}\right)\frac{e^{j\omega_E t}}{2} + \left(\frac{d\tilde{i}^*}{dt} - j\omega_E\tilde{i}^*\right)\frac{e^{-j\omega_E t}}{2} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \frac{d\tilde{i}^*}{dt} - j\omega_E\tilde{i}^* &= \tilde{q}_{tt}^* \\ \frac{d\tilde{i}}{dt} + j\omega_E\tilde{i} &= \tilde{q}_{tt} \end{aligned}\right.$$

Now by using e.g. a Bode plot [14,15], one can derive the following relationship between the square and the complex envelope of the electric current in the system's circuit.

$$i(t) = \frac{\tilde{i}e^{j\omega_E t} + \tilde{i}^* e^{-j\omega_E t}}{2} \Rightarrow i^2 = \frac{\overbrace{\tilde{i}^2 e^{j2\omega_E t} + \left(\tilde{i}^2\right)^* e^{-j2\omega_E t}}^{\text{High pass terms centered about } 2\omega_E} + 2\left(\tilde{i}e^{j\omega_E t}\right)\left(\tilde{i}^* e^{-j\omega_E t}\right)}{4} \Rightarrow$$
$$\Rightarrow i^2 \equiv \tfrac{1}{2}\left|\tilde{i}\right|^2, \text{ in the baseband where the mechanical subsystem operates}$$

This essentially expresses the physical fact that the magnetic force applied to the payload depends only on the power [25,26] injected to the system by the source driving the circuit and not e.g. the direction of the current. This fact, could lead to complications (e.g. undefined relative degree as analyzed in [9-19]) if one attempts applying exact feedback linearization directly to the entire system.

Finally, by employing the observation above in the baseband (i.e. LP [25-31]) mechanical subsystem dynamics, one obtains the following.

$$\begin{gathered} m\ddot{y} + b\dot{y} + ky = \frac{L_1}{2}i^2 + d \\ \Updownarrow \\ m\ddot{y} + b\dot{y} + ky = \frac{L_1}{4}\left|\tilde{i}\right|^2 + d \end{gathered} \qquad (1.102)$$

The above shows the importance and physical interpretation of both the LP equivalent and its value in the analysis of spectrally coupled subsystems through state modulation and demodulation. Also, as can be seen, exact feedback linearization in the modified fashion used here and introduced earlier can be a very useful tool in practical applications of e.g. driving electromechanical or mechatronic systems for Ocean Energy Harvesting applications.

As pointed out in the abstract of the joint work by Clauss, G. and Schmitz, R. [31]: "Hybrid structures in ocean engineering are based on flat concrete foundations. Due to wave action these foundations are exposed to different pressure distributions on top and bottom side… As a result the bottom side is exposed to a saddle type pressure distribution leading to huge forces on the foundation".

Indeed, such huge forces have been observed [31] at a number of offshore platforms installed in the North Sea, e.g. research platform Forschungsplattform Nordsee (Figure 3).

In an attempt to turn a problem into an advantage, the concept in this paper aims to develop an integrated system to harness and harvest ocean wave energy right at the seabed. The long-term interest is to develop integrated devices that can be used as actuators or sensors, which, due to low manufacturing cost, can be employed in large quantities for control of ocean engineering systems, e.g. maritime renewable power-plants, or monitoring of marine processes, e.g. oceanographic sensing.

Figure 3: Offshore platform Forschungsplattform Nordsee.

A key element to the proposed system is the nonlinear coupled electromechanical oscillator unit, the dynamics of which are investigated with a novel approach in this work. The fundamental nature of the oscillator at hand makes it an excellent choice for applications involving oceanic transducers consisting of a dry driving electrical stator physically separated from a wet driven payload mechanism [1,3]. Without such units available at a low cost and a large number, harvesting the energy of a vibrating plate at seabed may prove impractical.

Conclusions

This work investigated some aspects of nonlinear systems with state modulation and demodulation. As seen, even if such system consists of two spectrally decoupled subsystems for which no way exists to interact through linear dynamics, state modulation and demodulation may enable nonlinear coupling. In mechatronics and applications of Micro- and Nano-Electro-Mechanical Systems (MEMS/NEMS), modules with the general structure like the one presented here arise rather often. The capability of using different carrier frequencies to drive on the same line and possibly by the same source a series of transducers for sensing, actuation and control is critical for advanced instrumentation applications spanning diverse fields like maritime, automotive, aerospace etc.

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