Acta Scientific COMPUTER SCIENCES
Volume 4 Issue 8 August 2022

Research Article

Corrected Mathematical Model for the Top Motions with the Eccentric Mass

Ryspek Usubamatov*

Kyrgyz State Technical University After I. Razzakov, Bishkek, Kyrgyzstan

***Corresponding Author:** Ryspek Usubamatov, Kyrgyz State Technical University After I. Razzakov, Bishkek, Kyrgyzstan.

Received: June 05, 2022
Published: July 18, 2022

Abstract

The top toy is a simplest gyroscopic device which mathematical model for a motion is not well-presented in the known publications. Recent investigations have shown the gyroscope properties are more complex. The external force applied on a gyroscope generates of the interrelated centrifugal and Coriolis forces of the mass elements, the change in the angular momentum, and centre mass of the spinning rotor. These system of torques acting around two axes of the spinning top express the basic principles of the gyroscope theory. The system of the inertial torques formulate the mathematical models for the nutation of the top toy with the eccentric mass. This manuscript presents the mathematical model for the top nutation in terms of vibration analysis.

Keywords: Gyroscope Theory; Property; Top Nutation

Introduction

The top toy is the simplest gyroscope device because contains the spinning disc [1-4]. Classical mechanics consider gyroscopes with simplified theory that does not gave correct solutioning [5-8]. Complexity of action forces on the spinning disc and its intricated monitions with nutation are demonstrated as the physical problem in the textbooks [9,10]. One solution is presented if the nature of the top's nutation based on its center mass is eccentric. The action of the eccentric spinning mass generates a forced free oscillation that is the nutation and the typical vibration of dynamics analysis [11-13]. The forces and the torques acting on the top are described above and presented in the known publication [14].

The spinning top nutates because acts the eccentric mass. The top weight and interrelated inertial torques present a damper. The centrifugal force of the rotating eccentric mass manifests the shift of the top center mass and harmonic nutation. The shifted center mass rotates about the top axis, tilted top on the angle γ to the horizontal, by a sinus law y = esinα where e is the eccentricity of mass location [15]. For simplification is accepted the frictional torque acting on the top leg is neglected (Figure 1). The nutation of the top around two axes are decayed rapidly because of the action of the inertial torques that top turn it to the vertical. With a slowly rotating top, the nutation manifests a slow change of the top axial movement.

Recent research of gyroscopic problems gives the corrected inertial torques and their interrelation around two axes are presented in table 1 [16] and new phenomena of deactivation of inertial torques [17]. These new innovations are used for the mathematical modelling of the top motion.

Type of the torque generated by	Acton	Equation
Centrifugal forces, T_{ct}	Resistance	
	Precession	$T_{ct} = \frac{4}{9}\pi^2 J\boldsymbol{\omega}\,{}_x$
Coriolis forces, T_{cr}	Resistance	$T_{cr} = (8/9)J\boldsymbol{\omega}\,{}_x$
Change in angular momentum, T_{am}	Precession	$T_{am} = J\boldsymbol{\omega}\,{}_x$
Dependency of the angular velocities around two axes for horizontal disposition of the spinning disc		
$\omega_y = (8\pi^2 + 17)\omega_x$		

Table 1: Equations of the inertial torques acting on the spinning disc.

Where ω_i and ω is the angular velocity of gyroscope rotation around axis i and around axis oz, respectively; J is moment of inertia of the spinning disc.

Methodology

The equation of nutation is obtained one base of the action of its weigh and the inertial torques on the top.

The expressions of the inertial torques are presented in table 1. These forces and torques are demonstrated in figure 1. The expressions of the torques of the eccentric mass acting around axes ox and oy are as the following:

$$T_{ex} = Fl = Me\omega^2 l \sin\gamma \sin\alpha \text{ around axis ox} \quad \text{-------------- (1)}$$

$$T_{ey} = Fl = Me\omega^2 l \cos\gamma \cos\alpha \text{ around axis oy} \quad \text{-------------- (2)}$$

Where M and l is the mass and the length of the top leg, respectively, γ is the tilt angle of the top axis, $\alpha = 90^0$ is the angle used for calculating the maximal value (Section Introduction).

The torque produced by the weight W of a top is: $T = Wl = Mgl\cos\gamma$. The defined data use for the equations of top motions (Figure 1). The top motion does not consider the leg spiralled motion.

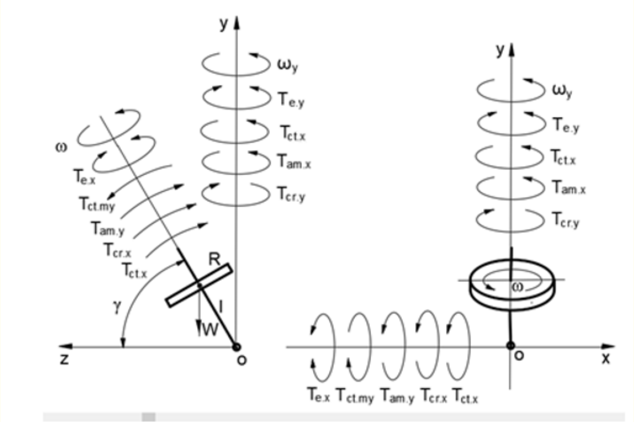

Figure 1: Force and torques acting on a spinning top with an shifted center mass.

The mathematical model for the motion of a top with the eccentric of mass around axes ox and oy is as the follows:

$$J_x \frac{d\omega_x}{dt} = T + T_{ct.my} \pm T_{ex} - T_{ctx} - T_{crx} - T_{amy} \quad \text{-------------(3)}$$

$$J_y \frac{d\omega_y}{dt} = \pm T_{ey} + T_{ct.x} + T_{amx} - T_{cry} \quad \text{------------- (4)}$$

Where $J_i = (MR^2/4) + Ml^2$ is the top mass moment of inertia around axis i, while the sign (±) denotes (+) and (-) as the positive and negative directions of the action, respectively.

The torques generated by the centrifugal forces of the rotating center mass about axis ox are defined by the following equation:

$T_{ct,my} = F_{ct.my} \times l\sin\gamma = Ml^2\cos\gamma \sin\gamma\, \omega_y^{\,2}$ ----------- (5)

Where $F_{ct.my} = Ml\cos\gamma\omega_y^{\,2}$ is the centrifugal force of the top center mass rotating about axis oy, and ω_y is the angular velocity of the top around axis oy; other components are as specified above.

Substituting expressions of inertial torques (Table1) corrected on the angle γ [16] into Eqs (3) and (4) and adding the dependency of the angular velocities of a top around axes $\omega_y = f(\omega_x)$ are yielded the equations of the top motions:

$$J_x \frac{d\omega_x}{dt} = Mgl\cos\gamma + Ml^2 \sin\gamma\cos\gamma\omega_y^2 \pm Me\omega^2 l \sin\gamma\sin\alpha - \left(\frac{4\pi^2+8}{9}\right) J\omega\omega_x - J\omega\omega_y \quad \text{---------(6)}$$

$$J_y \frac{d\omega_y}{dt} = \pm Me\omega^2 l\cos\alpha + \left(\frac{4\pi^2+9}{9}\right) J\omega\omega_x \cos\gamma - \frac{8}{9} J\omega\omega_y \quad \text{----------(7)}$$

$$\omega_y = -[4\pi^2 + 8 + (4\pi^2+9)\cos\gamma]\omega_x \quad \text{---------- (8)}$$

Where all parameters are as specified above.

Solving of Eqs. (6) - (8) yields the angular velocities ω_x and ω_y and the nutation amplitudes about axes, respectively. The amplitudes about axes ox and oy depict the trajectory of a top center mass nutation. The expression ω_y from Eq. (8) is substituted into Eq. (6) that yields the following:

$$J_x \frac{d\omega_x}{dt} = Mgl\cos\gamma + Ml^2\sin\gamma\cos\gamma \times [4\pi^2 + 8 + (4\pi^2+9)\cos\gamma]^2 \times \omega_x^2 \pm Me\omega^2 l\sin\gamma\sin\alpha - \left(\frac{4\pi^2+8}{9} + 4\pi^2 + 8 + [4\pi^2+9]\cos\gamma\right) J\omega\omega_x \quad \text{---------- (9)}$$

Where all parameters are as specified above.

Working example

The disc-type top with the eccentric of mass whose data is represented in table 2 is considered for the nutation process. The center mass is located at the plane of the disc (Figure 1). The action of the frictional force on the top leg is neglected. All technical data are the same as presented in my publication [15].

The top weight W produces the torque:

$T = Mgl\cos\gamma = 0{,}02\times9{,}81\times0{,}02\times\cos75{,}0^o = 1{,}015605932 \times 10^{-3}$ Nm --------- (10)

<table>
<tr><th colspan="2">Parameter</th><th>Data</th></tr>
<tr><td colspan="2">Angular velocity, ω</td><td>1000 rpm</td></tr>
<tr><td colspan="2">Radius of the disc, R</td><td>0,025m</td></tr>
<tr><td colspan="2">Length of the leg, l</td><td>0,02m</td></tr>
<tr><td colspan="2">The eccentricity of the center mass, e</td><td>0,001m</td></tr>
<tr><td colspan="2">Angle of tilt, γ</td><td>75,0°</td></tr>
<tr><td colspan="2">Mass, M</td><td>0,02 kg</td></tr>
<tr><td rowspan="3">Moment of inertia, kgm^2</td><td>About axis oz, $J = mR^2/2$</td><td>$0,625\times10^{-5}$</td></tr>
<tr><td>About axes ox and oy of the center mass, $J = mR^2/4$</td><td>$0,3125\times10^{-5}$</td></tr>
<tr><td>About axes ox and oy at the point of support, $J_x = J_y = mR^2/4 + ml^2$</td><td>$1,1125\times10^{-5}$</td></tr>
</table>

Table 2: Technical data of a top with the eccentric of mass.

The top center mass rotation about axis oy produces the centrifugal force:

$$T_{ct.my} = Ml^2 \cos\gamma \sin\gamma\omega_y^2 =$$
$$0,02\times0,02^2\times\cos 75^o \sin\gamma 75^o \times[4\pi^2+8+(4\pi^2+9)\cos 75^o]^2\times\omega_x^2$$
$$= 0,007206\times\omega_x^2 \quad Nm$$

---------(11)

The eccentric rotating center mass generates the maximal torque: about axis ox is as follows:

$$T_{ex} = \pm Me\omega^2 l\sin\gamma\sin\alpha = \pm 0,02\times0,001\times(1000\times2\pi/60)^2\times0,02\times$$
$$\sin 75^o\times\sin 90,0^0 = \pm 0,019318 Nm$$

---------(12)

Where all parameters are as specified above.

The defined parameters (Table 2) of Eqs. (10) - (12 is substituted into Eq. (9) and transforming yields:

$$1,1125\times10^{-5}\frac{d\omega_x}{dt} = 1,015605932\times10^{-3} + 0,007206\omega_x^2 \pm$$
$$0,019318 - 0,042739\omega_x$$

------------(13)

Simplification and transformation of Eq. (13) yield:

$$0,001543\frac{d\omega_x}{dt} = \omega_x^2 - 5,931029\omega_x + 2,821759 \quad \text{------------ (14)}$$

$$0,001543\frac{d\omega_x}{dt} = \omega_x^2 - 5,931029\omega_x - 2,539883 \quad \text{------------ (15)}$$

Separating variables and transformation of Eqs. (14) and (15) and presenting by the integral forms with defined limits yield:

$$\int_0^{\omega_x}\frac{d\omega_x}{\omega_x^2 - 5,931029\omega_x + 2,821759} = 648,088\int_0^t dt \quad \text{----------------- (16)}$$

$$\int_0^{\omega_x}\frac{d\omega_x}{\omega_x^2 - 5,931029\omega_x - 2,539883} = 648,088\int_0^t dt \quad \text{----------------- (17)}$$

The denominators of Eqs. (16) and (17) are the quadratic equation that resented by the following:

$$\int_0^{\omega_x}\frac{d\omega_x}{(\omega_x - 5,409387)(\omega_x - 0,521641)} = 648,088\int_0^t dt \quad \text{----------------- (18)}$$

$$\int_0^{\omega_x}\frac{d\omega_x}{(\omega_x - 6,332139)(\omega_x - 0,401109)} = 648,088\int_0^t dt \quad \text{----------------- (19)}$$

Integral Eqs. (18) and 19) are presented by the following:

$$\frac{1}{-4,887746}\int_0^{\omega_x}\left(\frac{1}{\omega_x - 0,521641} - \frac{1}{\omega_x - 5,409387}\right)d\omega_x = 648,088\int_0^t dt$$

-------------- (20)

$$\frac{1}{-6,733248}\int_0^{\omega_x}\left(\frac{1}{\omega_x + 0,401109} - \frac{1}{\omega_x - 6,332139}\right)d\omega_x = 648,088\int_0^t dt$$

-------------- (21)

The integrals of Eqs. (20) and (21) are tabulated and presented in the integral $\int\frac{dx}{x-a} = \ln|a-x| + C$ with the following solution:

$$\ln(\omega_x - 0,521641)\Big|_0^{\omega_x} - \ln(\omega_x - 5,409387)\Big|_0^{\omega_x} = -3167,689t\Big|_0^t$$

$$\ln(\omega_x + 0,401109)\Big|_0^{\omega_x} - \ln(\omega_x - \omega_x - 6,332139)\Big|_0^{\omega_x} = -4363,737t\Big|_0^t$$

That gave rise to the following:

$$\ln\left(\frac{5,409387}{0,521641}\times\frac{\omega_x - 0,521641}{\omega_x - 5,409387}\right) = -3167,689t \quad \text{-------------- (22)}$$

$$\ln\left(\frac{6,332139}{0,401109}\times\frac{\omega_x + 0,401109}{6,332139 - \omega_x}\right) = -4363,737t \quad \text{-------------- (21)}$$

The transformation yields the following result:

$$\omega_x - 0,521641 = 0,096432\times(\omega_x - 5,409387)e^{-3167,689t} \quad \text{----------- (22)}$$

$$\omega_x + 0,401109 = 0,063344\times(6,332139 - \omega_x)e^{-4363,737t} \quad \text{----------- (23)}$$

The right expressions of Eqs. (22) and (23) have a small value of a high order that is neglected. Equations (22) and (23) give the maximal and minimal value of the angular velocity for the top about axis ox:

$$\omega_{x.\max} = 0{,}521641 \quad rad/s \qquad \text{------------- (24)}$$

$$\omega_{x.\min} = -0{,}401109 \ \text{rad/s} \qquad \text{-------------- (25)}$$

Where the sign (-) means the turn of the top in the clockwise direction.

The maximal and minimal values of the linear velocity of the top center mass about axis ox are:

$$V_{y\max} = \omega_{x\max} l = 0{,}521641(rad/s) \times 20mm \times \cos 75^o = 2{,}700 \quad mm/s \qquad \text{----------- (26)}$$

$$V_{y\min} = \omega_{x\min} l = -0{,}401109(rad/s) \times 20mm \times \cos 75^o = -2{,}076 \quad mm/s \qquad \text{----------- (27)}$$

The maximal and minimal values of the amplitudes of center-mass nutation about axis ox are as follows:

$$a_{y\max} = V_{y\max} t = 2{,}700(mm/s) \times 0{,}03s = 0{,}081 \quad mm \qquad \text{----------- (28)}$$

$$a_{y\min} = V_{y\min} t = -2{,}076(mm/s) \times 0{,}03s = -0{,}062 \quad mm \qquad \text{------------ (29)}$$

The amplitude of oscillation about axis ox is:

$$a_y = a_{y\max} + |a_{y\min}| = 0{,}081 + 0{,}062 = 0{,}143 \quad mm \qquad \text{------------ (30)}$$

The weight of the top is F = Mg = 0,02×9,81 = 0,1962 N. The eccentric mass of the spinning top generates the centrifugal force $F_{ct} = Me\omega^2 = 0{,}02\times0{,}001\times(1000\times2\pi/60)^2 = 0{,}219$ N, shifts the top, and deactivates the inertial torques of the top [17]. The action of the force F_{ct} in the counter-clockwise direction (Figure 1) does not change the angular velocity of the top because the dependency of Eq. (8) is too big.

The amplitude of oscillation of the center mass around axis oy is defined by the angular velocity of the top.

$$J\frac{d\omega_y}{dt} = F_{ct} l \qquad \text{-------------- (31)}$$

Separating variables and integration yields

$$\int_0^{\omega_y} d\omega_y = \int_0^{0{,}03} \frac{F_{ct} l}{J} dt \qquad \text{------------ (32)}$$

Solving of Eq. (32) and substituting parameters defined above and $J = Ml^2/2 = 0{,}02\times0{,}02^2/2 = 4\times10^{-6}$ kgm^2 is the moment of inertia of the top about axis oy yields

$$\omega_y = \frac{F_{ct} l}{J} \times 0{,}03 = \frac{0{,}219 \times 0{,}02}{4\times10^{-6}} \times 0{,}03 = 32{,}85 \quad rad/s \qquad \text{------------ (33)}$$

The half rice amplitude of center mass nutation about axis oy is:

$$a_x = \omega_y l \sin\gamma \times t = 32{,}85 \times 0{,}02 \times \sin 75^o \times 0{,}03 = 0{,}019 \text{ mm} \qquad \text{-------------- (34)}$$

The total amplitude of oscillation about axis oy is:

$$a_x = a_x + |a_x| = 0{,}038 \quad mm \qquad \text{--------------- (35)}$$

The angular velocities about axes ox and oy (Eqs. (24), (25)), mean rotation in counter-clockwise and clockwise directions, respectively. The linear motions of the top center mass will be in negative and positive directions along the axis and represented in figure 2 with two coordinate systems. The system with asterisks shows the motions of a balanced top. Figure 2 shows the amplitudes for the top center mass have stretched down along the axis oy because the action of the centrifugal force and the top weight in one the direction. The opposite direction decreases the amplitude of nutation.

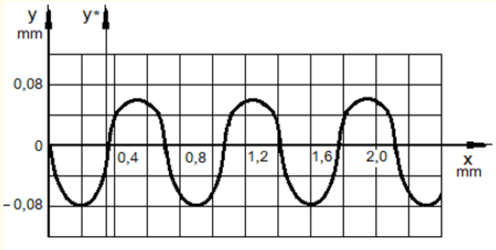

Figure 2: Deployed diagram of top nutation.

The diagram of top nutation is looped when the positive amplitude of oscillation along axis oy is less than the distance of top motion. Equations of top motions with nutation show the amplitude and frequency of nutation are cyclic. The cycles are different because the top eccentric mass acts at the first and the second semicircles in the opposite directions of precessed motions. The inertial torques of top acting about axis ox turns up the top to the vertical position. The analysis of the top nutation describes the physics of this process [13].

Results and Discussion

The mathematical model for nutation of the simplest top was presented by known publications in terms of the incorrect expressions of the centrifugal torque incorrect and the dependency of the angular velocities around two axes. This analytical study yields the incorrect motions of the top with the eccentricity of center-mass.

These models are based on action of several internal forces generated by the mass elements and center-mass of a top. The diagram of a top's nutation represents the motions of the top with mass eccentricity. The new mathematical model for the motions of the top with mass eccentricity under the action of the corrected inertial torques yields the new diagram of nutation.

Conclusion

In science, the top's nutation s was one of the most unsolvable analytically problem. The new mathematical approach for gyroscope torques gave the prospect for researchers but the science is the rode not only with the success but also with failures. Incorrect solutions are corrected and open the true knowledge in the science that happened with the inertial forces acting on the gyroscope. The corrected mathematical model for the centrifugal torque and the dependency of the angular velocities of the spinning disc around axes finally solved most gyroscopic problem. Among them was nutation of the top with the mass eccentricity. Solution of the top motion and nutation is primary task in engineering, but presents a good example for popularization and educational processes.

Bibliography

1. Armenise MN., *et al*. "Advances in Gyroscope Technologies". Springer-Verlag Berlin and Heidelberg GmbH and Co. KG, Berlin, (2010).
2. Deimel RF. "Mechanics of the Gyroscope." Dover Publications Inc. New York (2003).
3. Greenhill G. "Report on Gyroscopic Theory". General Books LLC, London (2010).
4. Scarborough JB. "The Gyroscope Theory and Applications". Nabu Press, London (2011).
5. Hibbeler RC. "Engineering Mechanics - Statics, and Dynamics". 12^{th} ed. Prentice-Hall, Pearson, Singapore, 2010.
6. Gregory DR. "Classical Mechanics". Cambridge University Press, New York, (2006).
7. Taylor JR. "Classical Mechanics". University Science Books, California, USA, (2005)
8. Aardema MD. "Analytical Dynamics". Theory and Application. Academic/Plenum Publishers, New York, (2005).
9. Jewett J and Serway RA. "Physics for Scientists and Engineers". Cengage Learning, 10 ed. USA, Boston, (2018).
10. Knight R D. "Physics for Scientists and Engineers: A Strategic Approach with Modern Physics". Pearson, 4 ed., UK, London, (2016).
11. Perry J. "Spinning Tops and Gyroscopic Motions". Literary Licensing, LLC, NV USA, (2012).
12. Klein F and Sommerfeld A. "The theory of the top. I – IV". New York, NY: Springer, Birkhäuser, (2008):2008-2014.
13. Usubamatov R. "Theory of gyroscopic effects for rotating objects". Springer, Singapore (2020).
14. Usubamatov R and Allen D. "Corrected Inertial Forces of Gyroscopic Effects". *Advances in Mathematical Physics* (2022): 7.
15. Usubamatov R and Omorova A. "A Mathematical Model for Top Nutation Based on Inertial Forces of Distributed Masses". *Mathematical Problems in Engineering* (2018).
16. Usubamatov R. "Corrected Internal Torques of Gyroscopic Effects". *Online Journal of Robotics and Automation Technology* 1.2 (2022):1-4.
17. Usubamatov R and Bergander M. "Physics of deactivation of gyroscopic inertial forces". *Journal of Mechanical Engineering Research* 12.1(2021): 30-36.