



# Recurrence Relations for Moments of Generalized Order Statistics from Length-biased Weighted Maxwell Distribution and its Characterization

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Received: January 11, 2022

Published: March 28, 2022

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### Abstract

In this paper, recurrence relations for single and product moments of generalized order statistics from length-biased weighted Maxwell distribution are obtained. Specializations to order statistics and records have been made. Further, using a recurrence relation for single moments we obtain a characterization of length-biased weighted Maxwell distribution.

**Keywords:** Generalized Order Statistics; Order Statistics; Records; Single and Product Moments; Recurrence Relations; Length-Biased; Weighted Maxwell Distribution; Characterization

### Introduction

A random sample X is said to have a Length-biased weighted Maxwell distribution (LBWMD) if its probability density function (pdf) is in the form:

$$f(x) = \frac{(\alpha-\beta)^2}{2} x^3 e^{-\frac{x^2}{2}(\alpha-\beta)}; \quad \beta > 0, \alpha > \beta \text{ and } x > 0 \tag{1}$$

The cumulative distribution function (CDF) and survival function (SF) are:

$$F(x) = 1 - \left[ \frac{x^2}{2}(\alpha-\beta) + 1 \right] e^{-\frac{x^2}{2}(\alpha-\beta)}; \quad \beta > 0, \alpha > \beta \text{ and } x > 0 \tag{2}$$

And

$$\bar{F}(x) = \left[ \frac{x^2}{2}(\alpha-\beta) + 1 \right] e^{-\frac{x^2}{2}(\alpha-\beta)}; \quad \beta > 0, \alpha > \beta \text{ and } x > 0 \tag{3}$$

Substituting from (3) in (1), we get

$$\bar{F}(x) = \left[ \frac{x^{-1}}{\alpha-\beta} + \frac{2x^{-3}}{(\alpha-\beta)^2} \right] f(x) \tag{4}$$

More details on this distribution and its applications can be found in Modi and Gill [18].

The concept of generalized order statistics (gos) was introduced by Kamps [12]. A variety of order models of random variables is contained in this concept.

Let, for simplicity,  $F$  throughout denote continuous distribution function with density function  $f$ . The random variables  $X(1, n, \tilde{m}, k), \dots, X(n, n, \tilde{m}, k)$  are called generalized order statistics based on  $F$ , if their joint pdf of the form

$$k \left( \prod_{j=1}^{n-1} \gamma_j \right) \left( \prod_{i=1}^{n-1} [\bar{F}(x_i)]^{m_i} f(x_i) \right) [\bar{F}(x_n)]^{k-1} f(x_n)$$

for,

$$F^{-1}(1) > x_1 \geq x_2 \geq \dots \geq x_n > F^{-1}(0)$$

with parameters  $n \in N, n \geq 2, k > 0,$

$$\tilde{m} = (m_1, m_2, \dots, m_{n-1}) \in R^{n-1}, M_r = \sum_{i=r}^{n-1} m_i,$$

such that  $\gamma_r = k + n - r + M_r > 0,$  for all  $r \in \{1, 2, \dots, n-1\}.$

For  $\gamma_i \neq \gamma_j, i \neq j$  for all  $i, j \in \{1, 2, \dots, n\}$  the pdf of  $X(r, n, \tilde{m}, k)$  is given by Cramer and Kamps [9] in the following way

$$f_{X(r, n, \tilde{m}, k)}(x) = C_{r-1} f(x) \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i - 1} \tag{5}$$

The joint pdf of  $X(r, n, \tilde{m}, k)$  and  $X(s, n, \tilde{m}, k)$   $1 \leq r < s \leq n$  is given as

$$f_{X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k)}(x, y) = C_{s-1} \left( \sum_{i=r+1}^s a_i^{(r)}(s) \left[ \frac{\bar{F}(y)}{\bar{F}(x)} \right]^{\gamma_i} \right) \left( \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} \right) \frac{f(x) f(y)}{\bar{F}(x) \bar{F}(y)}, \tag{6}$$

Where  $x < y$  and

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^r \frac{1}{\gamma_j - \gamma_i}, \quad 1 \leq i \leq r \leq n,$$

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \neq i}}^s \frac{1}{\gamma_j - \gamma_i}, \quad r+1 \leq i \leq s \leq n.$$

It may be noted that for  $m_1 = m_2 = \dots = m_{n-1} = m \neq -1,$

$$a_i(r) = \frac{(-1)^{r-i}}{(m+1)^{r-1} (r-1)!} \binom{r-1}{r-i}, \tag{7}$$

And

$$a_i^{(r)}(s) = \frac{(-1)^{s-i}}{(m+1)^{s-r-1} (s-r-1)!} \binom{s-r-1}{s-i}. \tag{8}$$

Therefore pdf of  $X(r, n, \tilde{m}, k)$  given in (5) reduces to

$$f_{X(r, n, \tilde{m}, k)}(x) = \frac{C_{r-1}}{(r-1)!} [\bar{F}(x)]^{\gamma_r - 1} f(x) g_m^{r-1}[F(x)], \tag{9}$$

$$f_{X(r, n, \tilde{m}, k), X(s, n, \tilde{m}, k)}(x, y) =$$

$$\frac{C_{s-1}}{(r-1)! (s-r-1)!} [\bar{F}(x)]^m f(x) g_m^{r-1}[F(x)] \{h_m[F(y)] - h_m[F(x)]\}^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(y), \quad x < y, \tag{10}$$

And joint pdf of  $X(r, n, \tilde{m}, k)$  and  $X(s, n, \tilde{m}, k)$  given in (6) reduces to  $\gamma_i = k + (n-i)(m+1)$

Where

$$C_{r-1} = \prod_{i=1}^r \gamma_i, \quad \gamma_i = k + (n-i)(m+1)$$

$$h_m(x) = \begin{cases} \frac{-1}{m+1} x^{m+1}, & m \neq -1 \\ -\ln x, & m = -1 \end{cases}$$

And

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0, 1)$$

We shall also take  $X(0, n, m, k) = 0$ . If  $m = 0, k = 1,$  then  $X(r, n, m, k)$  reduces to the  $(n-r+1)^{th}$  order statistics,  $X_{n-r+1:n}$  from the sample  $X_1, X_2, \dots, X_n$  and when  $m = -1,$  then  $X(r, n, m, k)$  reduces to the  $k^{th}$  record values (Pawlas and Szynal [19]).

Many authors utilized the (gos) in their work, such as Kamps and Gather [13], Keseling [14], Cramer and Kamps [9], Ahsanullah [5], Pawlas and Szynal [19], Ahmed [3], Ahmed and Fawzy [4], Khan., et al. [15], AL-Hussaini., et al. [6], Kumar [16], Mahmoud and Ghazal [17], Abdul-Moniem [1, 2], Athar., et al. [8] and Faten., et al. [10].

In this paper, we have established explicit expressions and some recurrence relations for single and product moments of (gos) LBWMD. Further its various deductions and particular cases are discussed. Characterization of LBWMD has been obtained by using a recurrence relation for single moments.

**Recurrence relation for single moments of (gos)**

$$\begin{aligned}
 & E[X^j(r, n, \tilde{m}, k)] \\
 &= \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}[F(x)] dx \\
 &= \frac{C_{r-1}}{(m+1)^{r-1} (r-1)!} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r-1} f(x) [1 - (F(x))^{m+1}]^{r-1} dx \\
 &= \frac{C_{r-1} \sum_{w=0}^{r-1} \binom{r-1}{w} (-1)^w}{(m+1)^{r-1} (r-1)!} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r-1} f(x) [F(x)]^w (m+1) dx \\
 &= \frac{C_{r-1} \sum_{w=0}^{r-1} \sum_{v=0}^{w(m+1)} \binom{r-1}{w} \binom{w(m+1)}{v} (-1)^{w+v}}{(m+1)^{r-1} (r-1)!} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r+v-1} f(x) dx \\
 &= \frac{j C_{r-1} \sum_{w=0}^{r-1} \sum_{v=0}^{w(m+1)} \binom{r-1}{w} \binom{w(m+1)}{v} (-1)^{w+v}}{(m+1)^{r-1} (r-1)! (\gamma_r+v)} \int_0^\infty x^{j-1} [\bar{F}(x)]^{\gamma_r+v} dx
 \end{aligned}$$

using (3), we get

$$\begin{aligned}
 & E[X^j(r, n, \tilde{m}, k)] = \\
 & \frac{j C_{r-1} \sum_{w=0}^{r-1} \sum_{v=0}^{w(m+1)} \binom{r-1}{w} \binom{w(m+1)}{v} (-1)^{w+v}}{(m+1)^{r-1} (r-1)! (\gamma_r+v)} \\
 & \int_0^\infty x^{j-1} \left[ \frac{x^2}{2} (\alpha-\beta)+1 \right]^{(\gamma_r+v)} e^{-\frac{x^2}{2} (\gamma_r+v) (\alpha-\beta)} dx \\
 &= \frac{j C_{r-1} \sum_{w=0}^{r-1} \sum_{v=0}^{w(m+1)} \sum_{\xi=0}^{\gamma_r+v} \binom{r-1}{w} \binom{w(m+1)}{v} \binom{\gamma_r+v}{\xi} (-1)^{w+v} \left( \frac{\alpha-\beta}{2} \right)^\xi}{(m+1)^{r-1} (r-1)! (\gamma_r+v)} \\
 & \int_0^\infty x^{j+2\xi-1} e^{-\frac{x^2}{2} (\gamma_r+v) (\alpha-\beta)} dx
 \end{aligned}$$

Let

$$y = \frac{x^2}{2} (\gamma_r+v) (\alpha-\beta) \Rightarrow x =$$

$$\sqrt{\frac{2y}{(\gamma_r+v) (\alpha-\beta)}} \text{ and } dx = \frac{y^{-\frac{1}{2}}}{2} \sqrt{\frac{2}{(\gamma_r+v) (\alpha-\beta)}} dy, \text{ we get}$$

$$\begin{aligned}
 & E[X^j(r, n, \tilde{m}, k)] = \\
 & \frac{j C_{r-1} \sum_{w=0}^{r-1} \sum_{v=0}^{w(m+1)} \sum_{\xi=0}^{\gamma_r+v} \binom{r-1}{w} \binom{w(m+1)}{v} \binom{\gamma_r+v}{\xi} (-1)^{w+v} 2^{\frac{j}{2}-1}}{(m+1)^{r-1} (r-1)! (\gamma_r+v)^{\frac{j}{2}+\xi+1} (\alpha-\beta)^{\frac{j}{2}}} \\
 & \int_0^\infty y^{\frac{j+2\xi-2}{2}} e^{-y} dy \\
 & \frac{j C_{r-1} \sum_{w=0}^{r-1} \sum_{v=0}^{w(m+1)} \sum_{\xi=0}^{\gamma_r+v} \binom{r-1}{w} \binom{w(m+1)}{v} \binom{\gamma_r+v}{\xi} (-1)^{w+v} 2^{\frac{j}{2}-1} \Gamma\left(\frac{j+2\xi}{2}\right)}{(m+1)^{r-1} (r-1)! (\gamma_r+v)^{\frac{j}{2}+\xi+1} (\alpha-\beta)^{\frac{j}{2}}}, \dots\dots (11)
 \end{aligned}$$

**Remark 2.1**

Putting  $m = 0, k = 1$  in (11), we obtain the single moments of order statistics as

$$\begin{aligned}
 & E[X_{r:n}^j] = \\
 & \frac{j n! \sum_{w=0}^{r-1} \sum_{v=0}^{n-r+v+1} \binom{r-1}{w} \binom{w}{v} \binom{n-r+v+1}{\xi} (-1)^{w+v} 2^{\frac{j}{2}-1} \Gamma\left(\frac{j+2\xi}{2}\right)}{(r-1)! (n-r)! (n-r+v+1)^{\frac{j}{2}+\xi+1} (\alpha-\beta)^{\frac{j}{2}}}
 \end{aligned}$$

**Theorem 2.1**

Let  $X$  be a random variable has pdf (1). Then for integer  $j$  such that  $j > 0$ , the following recurrence relation is satisfied.

$$\begin{aligned}
 & E[X^j(r, n, \tilde{m}, k)] - E[X^j(r-1, n, \tilde{m}, k)] = \\
 & \frac{j}{(\alpha-\beta)\gamma_r} E[X^{j-2}(r, n, \tilde{m}, k)] + \frac{2j}{(\alpha-\beta)^2\gamma_r} E[X^{j-4}(r, n, \tilde{m}, k)] \\
 & \dots\dots (12)
 \end{aligned}$$

**Proof**

We have from Lemma 2.3 (Athar and Islam [7]) that

$$\begin{aligned}
 & E[\xi \{X(r, n, \tilde{m}, k)\}] - E[\xi \{X(r-1, n, \tilde{m}, k)\}] = C_r \\
 & C_{r-2} \int_0^\beta \xi'(x) \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} dx
 \end{aligned}$$

If we let  $\xi(x) = x^j$ , then

$$E[X^j(r, n, \tilde{m}, k)] - E[X^j(r-1, n, \tilde{m}, k)] = \int_0^\beta j C_{r-2} x^{j-1} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i} dx \tag{13}$$

On using (4) in (13), we get

$$E[X^j(r, n, \tilde{m}, k)] - E[X^j(r-1, n, \tilde{m}, k)] = \frac{j C_{r-1}}{(\alpha - \beta) \gamma_r} \int_0^\infty x^{j-2} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} f(x) dx + \frac{2j C_{r-1}}{(\alpha - \beta)^2 \gamma_r} \int_0^\infty x^{j-4} \sum_{i=1}^r a_i(r) [\bar{F}(x)]^{\gamma_i-1} f(x) dx$$

Which after simplification leads to (12).□

**Corollary 2.2**

For  $m_1 = m_2 = \dots = m_{n-1} = m \neq -1$ , the recurrence relations for a single moment of (gos) for LBWMD is given as

$$E[X^j(r, n, m, k)] - E[X^j(r-1, n, m, k)] = \frac{j}{(\alpha - \beta) \gamma_r} E[X^{j-2}(r, n, m, k)] + \frac{2j}{(\alpha - \beta)^2 \gamma_r} E[X^{j-4}(r, n, m, k)] \tag{14}$$

**Proof**

This can easy be deduced from (12) given the relation (7).

**Remark 2.1**

Putting  $m = 0$ ,  $k = 1$  in Theorem 2.1., we obtain recurrence relations for single moments of order statistics as

$$E(X_{r:n}^j) - E(X_{r-1:n}^j) = \frac{j}{(\alpha - \beta)(n - r + 1)} \left[ E(X_{r:n}^{j-2}) + \frac{2}{(\alpha - \beta)} E(X_{r:n}^{j-4}) \right] \tag{15}$$

**Remark 2.2**

Setting  $m = -1$ ,  $k = 1$  in Theorem 2.1., we obtain the recurrence relations of upper record values as

$$E[X^j(r, n, -1, 1)] - E[X^j(r-1, n, -1, 1)] = \frac{j}{(\alpha - \beta)} \left\{ E[X^{j-2}(r, n, -1, 1)] + \frac{2}{(\alpha - \beta)} E[X^{j-4}(r, n, -1, 1)] \right\} \tag{16}$$

**Recurrence relation for product moments of (gos)**

**Theorem 3.1**

Let  $X$  be a random variable has pdf (1). Then for integer  $i, j$  such that  $i, j > 0$ , the following recurrence relation is satisfied.

$$E[X^i(r, n, \tilde{m}, k) X^j(s, n, \tilde{m}, k)] - E[X^i(r, n, \tilde{m}, k) X^j(s-1, n, \tilde{m}, k)] = \frac{j}{(\alpha - \beta) \gamma_s} E[X^i(r, n, \tilde{m}, k) X^{j-2}(s, n, \tilde{m}, k)] + \frac{2j}{(\alpha - \beta)^2 \gamma_s} E[X^i(r, n, \tilde{m}, k) X^{j-4}(s, n, \tilde{m}, k)] \tag{17}$$

**Proof**

We have from Lemma 3.2 (Athar and Islam [7]) that

$$E[\xi\{X^i(r, n, \tilde{m}, k) X^j(s, n, \tilde{m}, k)\}] - E[\xi\{X^i(r, n, \tilde{m}, k) X^j(s-1, n, \tilde{m}, k)\}] = C_{s-2} \int_0^\beta \int_0^\beta \frac{\partial}{\partial y} \xi(x, y) \sum_{l=r+1}^s a_l^{(r)}(s) \left[ \frac{\bar{F}(y)}{\bar{F}(x)} \right]^{\gamma_l} \sum_{l=1}^r a_l(r) [\bar{F}(x)]^{\gamma_l} \frac{f(x)}{\bar{F}(x)} dy dx$$

If we let  $\xi(x, y) = x^i y^j$ , then

$$E[X^i(r, n, \tilde{m}, k) X^j(s, n, \tilde{m}, k)] - E[X^i(r, n, \tilde{m}, k) X^j(s-1, n, \tilde{m}, k)] = \frac{j C_{s-1}}{\gamma_s} \int_0^\infty \int_0^\infty x^i y^{j-1} \sum_{l=r+1}^s a_l^{(r)}(s) \left[ \frac{\bar{F}(y)}{\bar{F}(x)} \right]^{\gamma_l} \sum_{l=1}^r a_l(r) [\bar{F}(x)]^{\gamma_l} \frac{f(x)}{\bar{F}(x)} dy dx$$

Given (4), note that

$$\frac{\bar{F}(y)}{f(y)} = \left[ \frac{y^{-1} + 2y^{-3}}{\alpha - \beta + (\alpha - \beta)^2} \right]$$

Therefore,

$$\begin{aligned} & E \left[ X^i(r, n, \tilde{m}, k) X^j(s, n, \tilde{m}, k) \right] - \\ & E \left[ X^i(r, n, \tilde{m}, k) X^j(s-1, n, \tilde{m}, k) \right] = \\ & \frac{j C_{s-1}}{\gamma_s} \int_0^\infty \int_0^\infty x^i y^{j-1} \left[ \frac{y^{-1} + 2y^{-3}}{\alpha - \beta + (\alpha - \beta)^2} \right] \sum_{l=r+1}^s a_l^{(r)}(s) \left[ \frac{\bar{F}(y)}{\bar{F}(x)} \right]^{\gamma_l} \\ & \sum_{l=1}^r a_l(r) [\bar{F}(x)]^{\gamma_l} \frac{f(x)f(y)}{\bar{F}(x)\bar{F}(y)} dy dx \\ & = \frac{j C_{s-1}}{(\alpha - \beta) \gamma_s} \int_0^\infty \int_0^\infty x^i y^{j-2} \sum_{l=r+1}^s a_l^{(r)}(s) \left[ \frac{\bar{F}(y)}{\bar{F}(x)} \right]^{\gamma_l} \\ & \sum_{l=1}^r a_l(r) [\bar{F}(x)]^{\gamma_l} \frac{f(x)f(y)}{\bar{F}(x)\bar{F}(y)} dy dx \\ & + \frac{2j C_{s-1}}{(\alpha - \beta)^2 \gamma_s} \int_0^\infty \int_0^\infty x^i y^{j-4} \sum_{l=r+1}^s a_l^{(r)}(s) \left[ \frac{\bar{F}(y)}{\bar{F}(x)} \right]^{\gamma_l} \\ & \sum_{l=1}^r a_l(r) [\bar{F}(x)]^{\gamma_l} \frac{f(x)f(y)}{\bar{F}(x)\bar{F}(y)} dy dx \end{aligned}$$

Which after simplification leads to (17).□

**Corollary 3.2**

For  $m_1 = m_2 = \dots = m_{n-1} = m \neq -1$ , the recurrence relations for product moments of gos for LBWMD is given as

$$\begin{aligned} & E \left[ X^i(r, n, m, k) X^j(s, n, m, k) \right] - \\ & E \left[ X^i(r, n, m, k) X^j(s-1, n, m, k) \right] \\ & = \frac{j}{(\alpha - \beta) \gamma_s} E \left[ X^i(r, n, m, k) X^{j-2}(s, n, m, k) \right] \\ & + \frac{2j}{(\alpha - \beta)^2 \gamma_s} E \left[ X^i(r, n, m, k) X^{j-4}(s, n, m, k) \right] \end{aligned} \quad \text{----- (18)}$$

**Proof**

This can easy be deduced from (17) given the relation (8).

**Remark 3.1**

Putting  $m = 0, k = 1$  in (18), we obtain recurrence relations for product moments of order statistics as

$$\begin{aligned} & E \left[ X_{r,s;n}^{i,j} \right] - E \left[ X_{r,s-1;n}^{i,j} \right] \\ & = \frac{j}{(\alpha - \beta)(n - s - 1)} \left\{ E \left[ X_{r,s;n}^{i,j-2} \right] + \frac{2}{(\alpha - \beta)} E \left[ X_{r,s;n}^{i,j-4} \right] \right\} \end{aligned} \quad \text{----- (19)}$$

**Remark 3.2**

Setting  $m = -1$  in (18), we obtain the recurrence relations for product moments of k<sup>th</sup> record values as

$$\begin{aligned} & E \left[ \left( X_r^{(k)} \right)^i \left( X_s^{(k)} \right)^j \right] - E \left[ \left( X_r^{(k)} \right)^i \left( X_{s-1}^{(k)} \right)^j \right] \\ & = \frac{j}{k(\alpha - \beta)} \left\{ E \left[ \left( X_r^{(k)} \right)^i \left( X_s^{(k)} \right)^{j-2} \right] + \frac{2}{(\alpha - \beta)} E \left[ \left( X_r^{(k)} \right)^i \left( X_s^{(k)} \right)^{j-4} \right] \right\} \end{aligned} \quad \text{----- (20)}$$

**Characterization**

**Theorem 4.1**

Let X be a non-negative random variable having an absolutely continuous distribution function F (x) with F (0) = 0 and 0 < F (x) < 1 for all x > 0, then

$$\begin{aligned} & E \left[ X^j(r, n, \tilde{m}, k) \right] - E \left[ X^j(r-1, n, \tilde{m}, k) \right] = \\ & \frac{j}{(\alpha - \beta) \gamma_r} E \left[ X^{j-2}(r, n, \tilde{m}, k) \right] + \frac{2j}{(\alpha - \beta)^2 \gamma_r} E \left[ X^{j-4}(r, n, \tilde{m}, k) \right] \end{aligned} \quad \text{----- (21)}$$

if and only if. 
$$\frac{\bar{F}(x)}{f(x)} = \left[ \frac{x^{-1} + 2x^{-3}}{\alpha - \beta + (\alpha - \beta)^2} \right]$$

**Proof**

The necessary part follows immediately from equation (12). On the other hand, if the recurrence relation in equation (21) is satisfied, then by using equation (5), we have

$$\begin{aligned} & \frac{C_{r-1}}{(r-1)!} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}[F(x)] dx - \\ & \frac{C_{r-2}}{(r-2)!} \int_0^\infty x^j [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-2}[F(x)] dx = \\ & \frac{jC_{r-1}}{(\alpha-\beta)\gamma_r(r-1)!} \int_0^\infty x^{j-2} [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}[F(x)] dx \\ & + \frac{2jC_{r-1}}{\gamma_r(\alpha-\beta)^2(r-1)!} \int_0^\infty x^{j-4} [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}[F(x)] dx \end{aligned}$$

Integrating the first term in left-hand side by parts, we get

$$\begin{aligned} & \frac{jC_{r-1}}{\gamma_r(r-1)!} \int_0^\infty x^{j-1} [\bar{F}(x)]^{\gamma_r} g_m^{r-1}[F(x)] dx = \\ & \frac{jC_{r-1}}{(\alpha-\beta)\gamma_r(r-1)!} \int_0^\infty x^{j-1} \left[ x^{-1} + \frac{2x^{-3}}{(\alpha-\beta)} \right] [\bar{F}(x)]^{\gamma_r-1} f(x) g_m^{r-1}[F(x)] dx \end{aligned}$$

This implies that

$$\begin{aligned} & \frac{jC_{r-1}}{\gamma_r(r-1)!} \int_0^\infty x^{j-1} [\bar{F}(x)]^{\gamma_r-1} g_m^{r-1}[F(x)] \cdot \\ & \left\{ \bar{F}(x) - \left[ \frac{x^{-1}}{\alpha-\beta} + \frac{2x^{-3}}{(\alpha-\beta)^2} \right] f(x) \right\} dx = 0 \end{aligned} \tag{22}$$

Now applying a generalization of the Muntz-Szasz theorem (Hwang and Lin [11]) to equation (22), we get

$$\frac{\bar{F}(x)}{f(x)} = \left[ \frac{x^{-1}}{\alpha-\beta} + \frac{2x^{-3}}{(\alpha-\beta)^2} \right]$$

### Conclusion

In this paper, we derive recurrence relations for single and product moments of generalized order statistics from length-biased weighted Maxwell distribution. We also studied specializations to order statistics and records. The characterization of length-biased weighted Maxwell distribution will be obtained using a recurrence relation for single moments.

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