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Review Article

Recurrence Relations for Moments of Generalized Order Statistics from Length-biased Weighted Maxwell Distribution and its Characterization

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Abstract

In this paper, recurrence relations for single and product moments of generalized order statistics from length-biased weighted Maxwell distribution are obtained. Specializations to order statistics and records have been made. Further, using a recurrence relation for single moments we obtain a characterization of length-biased weighted Maxwell distribution.

Keywords: Generalized Order Statistics; Order Statistics; Records; Single and Product Moments; Recurrence Relations; Length-Biased; Weighted Maxwell Distribution; Characterization

Introduction

A random sample X is said to have a Length-biased weighted Maxwell distribution (LBWMD) if its probability density function (pdf) is in the form:

$$f(x)=\frac{(\alpha-\beta)^2}{2}x^3 e^{-\frac{x^2}{2}(\alpha-\beta)}; \quad \beta>0, \alpha>\beta \text{ and } x>0 \qquad \text{---------- (1)}$$

The cumulative distribution function (CDF) and survival function (SF) are:

$$F(x)=1-\left[\frac{x^2}{2}(\alpha-\beta)+1\right]e^{-\frac{x^2}{2}(\alpha-\beta)}; \quad \beta>0, \alpha>\beta \text{ and } x>0 \qquad \text{-----------(2)}$$

And

$$\bar{F}(x)=\left[\frac{x^2}{2}(\alpha-\beta)+1\right]e^{-\frac{x^2}{2}(\alpha-\beta)}; \quad \beta>0, \alpha>\beta \text{ and } x>0 \qquad \text{-------------(3)}$$

Substituting from (3) in (1), we get

$$\bar{F}(x)=\left[\frac{x^{-1}}{\alpha-\beta}+\frac{2x^{-3}}{(\alpha-\beta)^2}\right]f(x) \qquad \text{---------------(4)}$$

More details on this distribution and its applications can be found in Modi and Gill [18].

The concept of generalized order statistics (gos) was introduced by Kamps [12]. A variety of order models of random variables is contained in this concept.

Let, for simplicity, F throughout denote continuous distribution function with density function f. The random variables $X(1,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)$ are called generalized order statistics based on F, if their joint pdf of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[\bar{F}(x_i)\right]^{m_i}f(x_i)\right)\left[\bar{F}(x_n)\right]^{k-1}f(x_n)$$

for,

$$F^{-1}(1)>x_1\geq x_2\geq\ldots\geq x_n>F^{-1}(0)$$

with parameters $n \in N$, $n \geq 2$, $k > 0$,

$\tilde{m} = (m_1, m_2, ..., m_{n-1}) \in R^{n-1}$, $M_r = \sum_{i=r}^{n-1} m_i$,

such that $\gamma_r = k + n - r + M_r > 0$, for all $r \in \{1,\ 2,\ ...,\ n-1\}$.

For $\gamma_i \neq \gamma_j, i \neq j$ for all $i, j \in (1, 2, ..., n)$ the pdf of $X(r, n, \tilde{m}, k)$ is given by Cramer and Kamps [9] in the following way

$$f_{X(r,n,\tilde{m},k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i - 1} \text{-----------(5)}$$

The joint pdf of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$ $1 \leq r < s \leq n$ is given as

$$f_{X(r,n,\tilde{m},k), X(s,n,\tilde{m},k)}(x, y) =$$

$$C_{s-1} \left(\sum_{i=r+1}^{s} a_i^{(r)}(s) \left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i} \right) \left(\sum_{i=1}^{r} a_i(r) \left[\bar{F}(x)\right]^{\gamma_i} \right) \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)},$$

---------(6)

Where $x < y$ and

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{\gamma_j - \gamma_i}, \quad 1 \leq i \leq r \leq n,$$

$$a_i^{(r)}(s) = \prod_{\substack{j=r+1 \\ j \neq i}}^{s} \frac{1}{\gamma_j - \gamma_i}, \quad r+1 \leq i \leq s \leq n.$$

It may be noted that for $m_1 = m_2 = ... = m_{n-1} = m \neq -1$,

$$a_i(r) = \frac{(-1)^{r-i}}{(m+1)^{r-1}(r-1)!} \binom{r-1}{r-i}, \qquad \text{-----------(7)}$$

And

$$a_i^{(r)}(s) = \frac{(-1)^{s-i}}{(m+1)^{s-r-1}(s-r-1)!} \binom{s-r-1}{s-i}. \qquad \text{------------(8)}$$

Therefore pdf of $X(r, n, \tilde{m}, k)$ given in (5) reduces to

$$f_{X(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!} \left[\bar{F}(x)\right]^{\gamma_r - 1} f(x) g_m^{r-1}\left[F(x)\right],$$

------------(9)

$$f_{X(r,n,m,k), X(s,n,m,k)}(x, y) =$$

$$\frac{C_{s-1}}{(r-1)!(s-r-1)!} \left[\bar{F}(x)\right]^{m} f(x) g_m^{r-1}\left[F(x)\right]$$

$$\left\{h_m\left[F(y)\right] - h_m\left[F(x)\right]\right\}^{s-r-1} \left[\bar{F}(y)\right]^{\gamma_s - 1} f(y), \qquad x < y$$

,------ (10)

And joint pdf of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$ given in (6) reduces to $\gamma_i = k + (n-i)(m+1)$

Where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1)$$

$$h_m(x) = \begin{cases} \frac{-1}{m+1} x^{m+1}, & m \neq -1 \\ -\ln x, & m = -1 \end{cases}$$

And

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0,\ 1)$$

We shall also take $X(0, n, m, k) = 0$. If m = 0, k = 1, then $X(r, n, m, k)$ reduces to the $(n-r+1)^{th}$ order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, ..., X_n$ and when m = -1, then $X(r, n, m, k)$ reduces to the kth record values (Pawlas and Szynal [19]).

Many authors utilized the (gos) in their work, such as Kamps and Gather [13], Keseling [14], Cramer and Kamps [9], Ahsanullah [5], Pawlas and Szynal [19], Ahmed [3], Ahmed and Fawzy [4], Khan., *et al*. [15], AL-Hussaini., *et al*. [6], Kumar [16], Mahmoud and Ghazal [17], Abdul-Moniem [1, 2], Athar., *et al*. [8] and Faten., *et al*. [10].

In this paper, we have established explicit expressions and some recurrence relations for single and product moments of (gos) LB-WMD. Further its various deductions and particular cases are discussed. Characterization of LBWMD has been obtained by using a recurrence relation for single moments.

Recurrence relation for single moments of (gos)

$$E\left[X^{j}\left(r,n,\tilde{m},k\right)\right]$$

$$=\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx$$

$$=\frac{C_{r-1}}{(m+1)^{r-1}(r-1)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)\left[1-\left(F(x)\right)^{m+1}\right]^{r-1}dx$$

$$=\frac{C_{r-1}\sum_{w=0}^{r-1}\binom{r-1}{w}(-1)^w}{(m+1)^{r-1}(r-1)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)\left[F(x)\right]^{w(m+1)}dx$$

$$=\frac{C_{r-1}\sum_{w=0}^{r-1}\sum_{v=0}^{w(m+1)}\binom{r-1}{w}\binom{w(m+1)}{v}(-1)^{w+v}}{(m+1)^{r-1}(r-1)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_r+v-1}f(x)dx$$

$$=\frac{jC_{r-1}\sum_{w=0}^{r-1}\sum_{v=0}^{w(m+1)}\binom{r-1}{w}\binom{w(m+1)}{v}(-1)^{w+v}}{(m+1)^{r-1}(r-1)!(\gamma_r+v)}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r+v}dx$$

using (3), we get

$$E\left[X^{j}\left(r,n,\tilde{m},k\right)\right]=$$

$$\frac{jC_{r-1}\sum_{w=0}^{r-1}\sum_{v=0}^{w(m+1)}\binom{r-1}{w}\binom{w(m+1)}{v}(-1)^{w+v}}{(m+1)^{r-1}(r-1)!(\gamma_r+v)}$$

$$\int_0^\infty x^{j-1}\left[\frac{x^2}{2}(\alpha-\beta)+1\right]^{(\gamma_r+v)}e^{-\frac{x^2}{2}(\gamma_r+v)(\alpha-\beta)}dx$$

$$=\frac{jC_{r-1}\sum_{w=0}^{r-1}\sum_{v=0}^{w(m+1)}\sum_{\xi=0}^{\gamma_r+v}\binom{r-1}{w}\binom{w(m+1)}{v}\binom{\gamma_r+v}{\xi}(-1)^{w+v}\left(\frac{\alpha-\beta}{2}\right)^{\xi}}{(m+1)^{r-1}(r-1)!(\gamma_r+v)}$$

$$\int_0^\infty x^{j+2\xi-1}e^{-\frac{x^2}{2}(\gamma_r+v)(\alpha-\beta)}dx$$

Let

$$y=\frac{x^2}{2}(\gamma_r+v)(\alpha-\beta)\Rightarrow x=$$

$\sqrt{\frac{2y}{(\gamma_r+v)(\alpha-\beta)}}$ and $dx=\frac{y^{-\frac{1}{2}}}{2}\sqrt{\frac{2}{(\gamma_r+v)(\alpha-\beta)}}dy$, we get

$$E\left[X^{j}\left(r,n,\tilde{m},k\right)\right]=$$

$$\frac{jC_{r-1}\sum_{w=0}^{r-1}\sum_{v=0}^{w(m+1)}\sum_{\xi=0}^{\gamma_r+v}\binom{r-1}{w}\binom{w(m+1)}{v}\binom{\gamma_r+v}{\xi}(-1)^{w+v}2^{\frac{j}{2}-1}}{(m+1)^{r-1}(r-1)!(\gamma_r+v)^{\frac{j}{2}+\xi+1}(\alpha-\beta)^{\frac{j}{2}}}$$

$$\int_0^\infty y^{\frac{j+2\xi-2}{2}}e^{-y}dy$$

$$\frac{jC_{r-1}\sum_{w=0}^{r-1}\sum_{v=0}^{w(m+1)}\sum_{\xi=0}^{\gamma_r+v}\binom{r-1}{w}\binom{w(m+1)}{v}\binom{\gamma_r+v}{\xi}(-1)^{w+v}2^{\frac{j}{2}-1}\Gamma\left(\frac{j+2\xi}{2}\right)}{(m+1)^{r-1}(r-1)!(\gamma_r+v)^{\frac{j}{2}+\xi+1}(\alpha-\beta)^{\frac{j}{2}}}$$

,------ (11)

Remark 2.1

Putting $m=0,\ k=1$ in (11), we obtain the single moments of order statistics as

$$E\left[X_{r:n}^{j}\right]=$$

$$\frac{jn!\sum_{w=0}^{r-1}\sum_{v=0}^{w}\sum_{\xi=0}^{n-r+v+1}\binom{r-1}{w}\binom{w}{v}\binom{n-r+v+1}{\xi}(-1)^{w+v}2^{\frac{j}{2}-1}\Gamma\left(\frac{j+2\xi}{2}\right)}{(r-1)!(n-r)!(n-r+v+1)^{\frac{j}{2}+\xi+1}(\alpha-\beta)^{\frac{j}{2}}}$$

Theorem 2.1

Let X be a random variable has pdf (1). Then for integer j such that $j>0$, the following recurrence relation is satisfied.

$$E\left[X^{j}\left(r,n,\tilde{m},k\right)\right]-E\left[X^{j}\left(r-1,n,\tilde{m},k\right)\right]=$$

$$\frac{j}{(\alpha-\beta)\gamma_r}E\left[X^{j-2}\left(r,n,\tilde{m},k\right)\right]+\frac{2j}{(\alpha-\beta)^2\gamma_r}E\left[X^{j-4}\left(r,n,\tilde{m},k\right)\right]$$

,------ (12)

Proof

We have from Lemma 2.3 (Athar and Islam [7]) that

$$E\left[\xi\left\{X\left(r,n,\tilde{m},k\right)\right\}\right]-E\left[\xi\left\{X\left(r-1,n,\tilde{m},k\right)\right\}\right]=C_r$$

$$C_{r-2}\int_\theta^\beta \xi'(x)\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i}dx$$

If we let $\xi(x)=x^{j}$, then

$$E\left[X^{j}(r,n,\tilde{m},k)\right]-E\left[X^{j}(r-1,n,\tilde{m},k)\right]=$$

$$jC_{r-2}\int_{\theta}^{\beta}x^{j-1}\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i}dx \qquad \text{,------ (13)}$$

On using (4) in (13), we get

$$E\left[X^{j}(r,n,\tilde{m},k)\right]-E\left[X^{j}(r-1,n,\tilde{m},k)\right]$$

$$=\frac{jC_{r-1}}{(\alpha-\beta)\gamma_r}\int_{0}^{\infty}x^{j-2}\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i-1}f(x)dx$$

$$+\frac{2jC_{r-1}}{(\alpha-\beta)^2\gamma_r}\int_{0}^{\infty}x^{j-4}\sum_{i=1}^{r}a_i(r)\left[\bar{F}(x)\right]^{\gamma_i-1}f(x)dx$$

Which after simplification leads to (12).□

Corollary 2.2

For $m_1=m_2=\ldots=m_{n-1}=m\neq-1,$ the recurrence relations for a single moment of (gos) for LBWMD is given as

$$E\left[X^{j}(r,n,m,k)\right]-E\left[X^{j}(r-1,n,m,k)\right]=$$

$$\frac{j}{(\alpha-\beta)\gamma_r}E\left[X^{j-2}(r,n,m,k)\right]+\frac{2j}{(\alpha-\beta)^2\gamma_r}E\left[X^{j-4}(r,n,m,k)\right] \qquad \text{,------ (14)}$$

Proof

This can easy be deduced from (12) given the relation (7).

Remark 2.1

Putting $m=0,\quad k=1$ in Theorem 2.1., we obtain recurrence relations for single moments of order statistics as

$$E\left(X_{r:n}^{j}\right)-E\left(X_{r-1:n}^{j}\right)=$$

$$\frac{j}{(\alpha-\beta)(n-r+1)}\left[E\left(X_{r:n}^{j-2}\right)+\frac{2}{(\alpha-\beta)}E\left(X_{r:n}^{j-4}\right)\right] \qquad \text{------ (15)}$$

Remark 2.2

Setting $m=-1,\ k=1$ in Theorem 2.1., we obtain the recurrence relations of upper record values as

$$E\left[X^{j}(r,n,-1,1)\right]-E\left[X^{j}(r-1,n,-1,1)\right]$$

$$=\frac{j}{(\alpha-\beta)}\left\{E\left[X^{j-2}(r,n,-1,1)\right]+\frac{2}{(\alpha-\beta)}E\left[X^{j-4}(r,n,-1,1)\right]\right\} \qquad \text{------ (16)}$$

Recurrence relation for product moments of (gos)

Theorem 3.1

Let X be a random variable has pdf (1). Then for integer i,j such that $i,j>0$, the following recurrence relation is satisfied.

$$E\left[X^{i}(r,n,\tilde{m},k).X^{j}(s,n,\tilde{m},k)\right]-$$

$$E\left[X^{i}(r,n,\tilde{m},k).X^{j}(s-1,n,\tilde{m},k)\right]$$

$$=\frac{j}{(\alpha-\beta)\gamma_s}E\left[X^{i}(r,n,\tilde{m},k)X^{j-2}(s,n,\tilde{m},k)\right]$$

$$+\frac{2j}{(\alpha-\beta)^2\gamma_s}E\left[X^{i}(r,n,\tilde{m},k)X^{j-4}(s,n,\tilde{m},k)\right] \qquad \text{------ (17)}$$

Proof

We have from Lemma 3.2 (Athar and Islam [7]) that

$$E\left[\xi\left\{X^{i}(r,n,\tilde{m},k).X^{j}(s,n,\tilde{m},k)\right\}\right]-$$

$$E\left[\xi\left\{X^{i}(r,n,\tilde{m},k).X^{j}(s-1,n,\tilde{m},k)\right\}\right]=$$

$$C_{s-2}\int_{\theta}^{\beta}\int_{x}^{\beta}\frac{\partial}{\partial y}\xi(x,y)\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_l}\sum_{l=1}^{r}a_l(r)\left[\bar{F}(x)\right]^{\gamma_l}\frac{f(x)}{\bar{F}(x)}dydx$$

If we let $\xi(x,y)=x^{i}y^{j}$, then

$$E\left[X^{i}(r,n,\tilde{m},k).X^{j}(s,n,\tilde{m},k)\right]-$$

$$E\left[X^{i}(r,n,\tilde{m},k).X^{j}(s-1,n,\tilde{m},k)\right]=$$

$$\frac{jC_{s-1}}{\gamma_s}\int_{0}^{\infty}\int_{x}^{\infty}x^{i}y^{j-1}\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_l}\sum_{l=1}^{r}a_l(r)\left[\bar{F}(x)\right]^{\gamma_l}\frac{f(x)}{\bar{F}(x)}dydx$$

Given (4), note that

$$\frac{\bar{F}(y)}{f(y)}=\left[\frac{y^{-1}}{\alpha-\beta}+\frac{2y^{-3}}{(\alpha-\beta)^2}\right]$$

Therefore,

$$E\left[X^{i}\left(r,n,\tilde{m},k\right).X^{j}\left(s,n,\tilde{m},k\right)\right]-$$

$$E\left[X^{i}\left(r,n,\tilde{m},k\right).X^{j}\left(s-1,n,\tilde{m},k\right)\right]=$$

$$\frac{jC_{s-1}}{\gamma_s}\int_0^\infty\int_x^\infty x^i y^{j-1}\left[\frac{y^{-1}}{\alpha-\beta}+\frac{2y^{-3}}{(\alpha-\beta)^2}\right]\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_l}$$

$$\sum_{l=1}^{r}a_l(r)\left[\bar{F}(x)\right]^{\gamma_l}\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx$$

$$=\frac{jC_{s-1}}{(\alpha-\beta)\gamma_s}\int_0^\infty\int_x^\infty x^i y^{j-2}\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_l}$$

$$\sum_{l=1}^{r}a_l(r)\left[\bar{F}(x)\right]^{\gamma_l}\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx$$

$$+\frac{2jC_{s-1}}{(\alpha-\beta)^2\gamma_s}\int_0^\infty\int_x^\infty x^i y^{j-4}\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_l}$$

$$\sum_{l=1}^{r}a_l(r)\left[\bar{F}(x)\right]^{\gamma_l}\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx$$

Which after simplification leads to (17).□

Corollary 3.2

For $m_1=m_2=\ldots=m_{n-1}=m\neq-1,$ the recurrence relations for product moments of gos for LBWMD is given as

$$E\left[X^{i}\left(r,n,m,k\right).X^{j}\left(s,n,m,k\right)\right]-$$

$$E\left[X^{i}\left(r,n,m,k\right).X^{j}\left(s-1,n,m,k\right)\right]$$

$$=\frac{j}{(\alpha-\beta)\gamma_s}E\left[X^{i}\left(r,n,m,k\right)X^{j-2}\left(s,n,m,k\right)\right]$$

$$+\frac{2j}{(\alpha-\beta)^2\gamma_s}E\left[X^{i}\left(r,n,m,k\right)X^{j-4}\left(s,n,m,k\right)\right]$$

------ (18)

Proof

This can easy be deduced from (17) given the relation (8).

Remark 3.1

Putting $m=0,\ k=1$ in (18), we obtain recurrence relations for product moments of order statistics as

$$E\left[X_{r,s:n}^{i,j}\right]-E\left[X_{r,s-1:n}^{i,j}\right]$$

$$=\frac{j}{(\alpha-\beta)(n-s-1)}\left\{E\left[X_{r,s:n}^{i,j-2}\right]+\frac{2}{(\alpha-\beta)}E\left[X_{r,s:n}^{i,j-4}\right]\right\}$$

------ (19)

Remark 3.2

Setting $m=-1$ in (18), we obtain the recurrence relations for product moments of k[th] record values as

$$E\left[\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^j\right]-E\left[\left(X_r^{(k)}\right)^i\left(X_{s-1}^{(k)}\right)^j\right]$$

$$=\frac{j}{k(\alpha-\beta)}\left\{E\left[\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^{j-2}\right]+\frac{2}{(\alpha-\beta)}E\left[\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^{j-4}\right]\right\}$$

------ (20)

Characterization

Theorem 4.1

Let X be a non-negative random variable having an absolutely continuous distribution function F (x) with F (0) = 0 and 0 < F (x) < 1 for all x > 0, then

$$E\left[X^{j}\left(r,n,\tilde{m},k\right)\right]-E\left[X^{j}\left(r-1,n,\tilde{m},k\right)\right]=$$

$$\frac{j}{(\alpha-\beta)\gamma_r}E\left[X^{j-2}\left(r,n,\tilde{m},k\right)\right]+\frac{2j}{(\alpha-\beta)^2\gamma_r}E\left[X^{j-4}\left(r,n,\tilde{m},k\right)\right]$$

------ (21)

if and only if. $\frac{\bar{F}(x)}{f(x)}=\left[\frac{x^{-1}}{\alpha-\beta}+\frac{2x^{-3}}{(\alpha-\beta)^2}\right]$

Proof

The necessary part follows immediately from equation (12). On the other hand, if the recurrence relation in equation (21) is satisfied, then by using equation (5), we have

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx-$$

$$\frac{C_{r-2}}{(r-2)!}\int_0^\infty x^j\left[\bar{F}(x)\right]^{\gamma_{r-1}-1}f(x)g_m^{r-2}\left[F(x)\right]dx=$$

$$\frac{jC_{r-1}}{(\alpha-\beta)\gamma_r(r-1)!}\int_0^\infty x^{j-2}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx$$

$$+\frac{2jC_{r-1}}{\gamma_r(\alpha-\beta)^2(r-1)!}\int_0^\infty x^{j-4}\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx$$

Integrating the first term in left-hand side by parts, we get

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r}g_m^{r-1}\left[F(x)\right]dx=$$

$$\frac{jC_{r-1}}{(\alpha-\beta)\gamma_r(r-1)!}\int_0^\infty x^{j-1}\left[x^{-1}+\frac{2x^{-3}}{(\alpha-\beta)}\right]\left[\bar{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right]dx$$

This implies that

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r-1}g_m^{r-1}\left[F(x)\right]\cdot$$

$$\left\{\bar{F}(x)-\left[\frac{x^{-1}}{\alpha-\beta}+\frac{2x^{-3}}{(\alpha-\beta)^2}\right]f(x)\right\}dx=0 \qquad \text{------ (22)}$$

Now applying a generalization of the Muntz-Szasz theorem (Hwang and Lin [11]) to equation (22), we get

$$\frac{\bar{F}(x)}{f(x)}=\left[\frac{x^{-1}}{\alpha-\beta}+\frac{2x^{-3}}{(\alpha-\beta)^2}\right]$$

Conclusion

In this paper, we derive recurrence relations for single and product moments of generalized order statistics from length-biased weighted Maxwell distribution. We also studied specializations to order statistics and records. The characterization of length-biased weighted Maxwell distribution will be obtained using a recurrence relation for single moments.

Bibliography

1. IB Abdul-Moniem. “Recurrence Relations for Moments of Generalized Order Statistics from Marshall - Olkin extended Kumaraswamy distribution and its Characterization”. *American Journal of Applied Mathematics and Statistics* 2.5 (2014): 324-329.

2. IB Abdul-Moniem. “Recurrence Relations for Moments of Generalized Order Statistics from Marshall - Olkin Extended Family of Life Distributions and its Characterization”. *Sohag Journal of Mathematics* 6.3 (2019): 45-50.

3. AA Ahmad. “Recurrence relations for single and product moments of generalized order statistics from doubly truncated Burr type XII distribution”. *Journal of the Egyptian Mathematical Society* 15.1 (2007): 117-128.

4. AA Ahmad and AM Fawzy. “Recurrence relations for single moments of generalized order statistics from doubly truncated distribution”. *Journal of Statistical Planning and Inference* 117.2 (2003): 241-249.

5. M Ahsanullah. “Generalized order statistics from exponential distribution”. *Journal of Statistical Planning and Inference* 85.1-2 (2000): 85-91.

6. EK AL-Hussaini., *et al*. “Recurrence relations for moment and conditional moment generating functions of generalized order statistics”. *Metrika* 61.2 (2005): 199-220.

7. H Athar and H Islam. “Recurrence relations for single and product moments of generalized order statistics from a general class of distribution”. *Metron* LXII.3 (2004): 327-337.

8. H Athar., *et al*. “Expectation Properties of Generalized Order Statistics from Poisson Lomax Distribution”. *Statistics, Optimization and Information Computing* 9 (2021): 735-747.

9. E Cramer and U Kamps. “Relations for expectations of functions of generalized order statistics”. *Journal of Statistical Planning and Inference* 89.1-2 (2000): 79-89.

10. A Faten., *et al*. “On Relations for Moments of Generalized Order Statistics for Lindley-Weibull Distribution”. *CSSE* 41.1 (2022): 197-208.

11. JS Hwang and GD Lin. "On a Generalized Moment Problem II". *Proceedings of the American Mathematical Society* 91 (1984): 577-580.

12. U Kamps. "A Concept of Generalized Order Statistics". B. G. Teubner, Stuttgart, (1995).

13. U Kamps and U Gather. "Characteristic property of generalized order statistics for exponential distribution". *Applied Mathematics (Warsaw)* 24 (1997): 383-391.

14. C. Keseling. "Conditional distributions of generalized order statistics and some characterizations". *Metrika* 49.1 (1999): 27-40.

15. R U Khan., *et al*. "Recurrence relations for single and product moments of generalized order statistics from doubly truncated Weibull distribution". *Aligarh Journal of Statistics* 27 (2007): 69-79.

16. D Kumar. "Generalized order statistics from Kumaraswamy distribution and its characterization". *Tamsui Oxford Journal of Information and Mathematical Sciences* 27.4 (2011): 463-476.

17. MAW Mahmoud and MGM Ghazal. "Characterization of exponentiated family of distributions based on recurrence relations for generalized order statistics". *Journal of Mathematics and Computer Science* 2.6 (2012): 1894-1908.

18. K Modi and V Gill. "Length-biased weighted Maxwell distribution". *Pakistan Journal of Statistics and Operation Research* 11 (2015): 465-472.

19. P Pawlas and D Szynal. "Recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto and Burr distributions". *Communications in Statistics - Theory and Methods* 30.4 (2001): 739-746.