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Review Article

# Approximate Analytical Solution of the Problem of the Theoretical Profile of Dimensionless Velocity in the Thickness of the Boundary Layer with Turbulent Flow in the Boundary Layer Based on the Solution of the Abel Differential Equation of the Second Kind by the Method of Successive Approximations with Additional Assumptions

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## Abstract

In this scientific article, approximate solutions were implemented for theoretical profiles for dimensionless velocities in the thickness of boundary layers for turbulent flows in boundary layers, which are based on the exact analytical solutions of differential equations for tangential stresses in turbulent boundary layers obtained earlier by the author of the article, which in turn are special cases of Abelian ordinary differential equations of the second order of the second kind using Lambert special function that have no solutions in quadrature's.

**Keywords:** Theoretical; Modeling; Mathematical; Approximate; Velocity; Method of Sequential Approximation; Coordinate; Dimensionless; Profile; Heat Transfer; Turbulent; Flow; Boundary Layer; Abelian Differential Equation; Second Order; Second Kind; Lambert Function

## Introduction

The determination of regularities about velocity profiles in planar turbulent flows for incompressible gases and liquids can be realized either as a result of theoretical analysis, or by introducing either experimental or partial experimental regularities.

Experimental and partially experimental regularities on the velocity distribution in turbulent boundary layers are relatively numerous [1-3], for example, in Loitsyansky, Reithard, Dysler, Van Dreitst, etc.

Velocity profiling in the vicinity of the combined effects of turbulent and molecular viscosities is a composite multi-configuration that includes piecewise functional continuities for a considerable number of individual segments.

Von Karman, for practical reasons, postulated stratifying the boundary layer with triple zonality when approximating zones, except for one, with dependencies based on a logarithmic representation.

It is possible to replace the multifunctional dependence of velocity profiling in turbulent flows with dependencies based on a power function, in which velocity profiles based on logarithms are envelopes for aggregates of power profiling.

Logarithmic velocity profiling alone can be an interpretation of the actual existence of a multifunctional pattern of relative velocity locations in turbulent boundary layers during the flow of turbulent isothermal free flows of incompressible gases or liquids surrounding spaces of impermeable planes [1-3].

## Mathematical model for the boundary layer introduction

The bases for determining approximate analytical solutions to the problem of theoretical profiles of relative velocities as a function of the thickness of boundary layers in turbulent flows in boundary layers based on the solution of Abelian ordinary differential equations of second orders and second genera, a method of sequential approximation using specific assumptions are scientific articles [4,5], in which analytical formulas for solving the problem of theoretical velocity profiling by the thickness of boundary layers with turbulent flows in the above-mentioned boundary layers were generated on the basis of the obtained solutions of the above Abelian ordinary differential equations of the second orders of the second kind using the Lambert special function.

To understand the method of implementing approximate analytical solutions to this issue, it is necessary to very briefly outline the basic aspects of analytical solutions [4,5], since the approximate analytical solution obtained in this article correlates with the solution previously implemented in [4,5].

In this article, the formulation of the problem for approximate theoretical solutions with respect to velocity profiling in planar turbulent boundary layers is based on the results of solutions of ordinary differential equations for tangential stress, which is justified in [6].

Tangential total stresses τ are calculated based on the following addition of individual stress components [6]:

$$\tau=\mu\frac{dw}{dy}+\rho\psi\zeta\left(l\frac{dw}{dy}+\frac{1}{2}\frac{d^2w}{dy^2}l^2\right)^2, \text{ -------------- (1)}$$

Where μ — are dynamic viscosities; w — are longitudinal velocities; y — are transverse coordinates; ρ are densities; l — are single-line characteristics like the parameter of mixing paths; ζ — are substitution coefficients; ψ — are correlation coefficients.

In the future, equation (1) should be dimensioned by entering relative coordinates $\eta = yw_*/\nu$ and relative velocities $\varphi = w/w_*$ (ν=μ/ρ — kinematic viscosities; $w_*=\sqrt{\tau_w/\rho}$ — «friction velocities»):

$$\frac{d\varphi}{d\eta}+\frac{\kappa^3}{\sqrt{\psi\zeta}}\eta^3\frac{d\varphi}{d\eta}\frac{d^2\varphi}{d\eta^2}=1 \text{ . --------------- (2)}$$

The last equation is an Abelian ordinary differential equation of the 2nd order of the 2nd kind (a special case) [6], unsolvable in quadrature [6].

Later in [4,5], solutions of this equation were obtained using the Lambert special function [7] under the following boundary conditions (asymptotic boundary conditions):

$$\eta\to 0:\ \frac{d\varphi}{d\eta}\to\frac{1}{1+(n\eta)^4}\approx 1-(n\eta)^4 \text{ , --------------- (3)}$$

Where n = 0,124;

$$\eta>>0:\ \frac{d\varphi}{d\eta}\to\frac{1}{\varkappa\eta}, \text{ ---------------- (4)}$$

Where = 0,4.

These solutions are for theoretical profiles of dimensionless velocities in the thickness of boundary layers, according to [4,5] and with partial application [6-9], it can be fixed in the following way:

$$\varphi(\eta)=\int_0^\eta W\left(-\frac{\kappa\eta}{(\kappa\eta-1)e^{\frac{\kappa\eta+1}{\kappa\eta}}}\right)d\eta+\eta. \text{ ----------- (5)}$$

The articles [4,5] also analyze the obtained analytical theoretical solutions, identify their advantages and disadvantages. For example, in [4,5], a certain contradiction was revealed regarding the functional nature for the boundary conditions that were used in the study [6].

The previously obtained analytical solution (5) of equation (2) leads to quadratures of special functions, so there is an interest in an approximate solution of this equation by a sequential approximation by differentiation.

This method has some advantages in solving this equation, which describes a physical phenomenon, but in some cases the method can lead to divergent solutions.

We introduce the following notation:

$$f(\eta) = \frac{\kappa^3}{\sqrt{\psi\zeta}}. \text{ --------------- (6)}$$

In the first approximation, we can put (κ' - constant) that:

$$\frac{d\varphi}{d\eta} = \frac{1}{\kappa'}\frac{1}{\eta}. \text{ --------------- (7)}$$

Differentiating (7) by n, we get differentiating (7) by η, we get:

$$\frac{d^2\varphi}{d\eta^2} = -\frac{1}{\kappa'}\frac{1}{\eta^2}. \text{ ----------- (8)}$$

If we substitute (7) and (8) into equation (2), then we can make sure that the term determining tangential stresses in this case has become negative. The latter contradicts the physical meaning, so for this technique it should be changed to positive. In this case, there is a disadvantage directly in the formulation of the problem being solved.

With the direct numerical solution of equation (2) above, the detected drawback of the model will not be revealed.

If we substitute positive expressions for derivatives (7) and (8) in (2), then we can get the following:

$$\frac{d\varphi}{d\eta} = \frac{1}{1+f(\eta)\frac{\eta}{\kappa'}}. \text{ --------------(9)}$$

Now we should get the second positive derivative $\frac{d^2\varphi}{d\eta^2}$ by differentiating the expression (9):

$$\frac{d^2\varphi}{d\eta^2} = \frac{\eta\frac{df(\eta)}{d\eta}+f(\eta)}{\kappa'\left[1+f(\eta)\frac{\eta}{\kappa'}\right]^2}. \text{ --------------------- (10)}$$

The following approximation is obtained by substituting (9) and (10) into equation (2):

$$\frac{d\varphi}{d\eta} = \frac{1}{1+\frac{f(\eta)}{\kappa'}\frac{\eta\frac{df(\eta)}{d\eta}+f(\eta)}{\left[1+f(\eta)\frac{\eta}{\kappa'}\right]^2}\eta^3}. \text{ ------------------------(11)}$$

The last expression can be limited, after which it is necessary to find the function f(η) and the constant κ', based on the asymptotic boundary conditions (3) and (4), and by hypothesizing (a=const):

$$f(\eta) = a(f(\eta))^{1/2}. \text{ --------------------(12)}$$

Substituting (12) into (11), we get:

$$\frac{d\varphi}{d\eta} = \frac{1}{1+\frac{\frac{3}{2}\frac{a^2}{\kappa'}\eta^4}{\left[1+\frac{a}{\kappa'}\eta^{3/2}\right]^2}}. \text{ -------------(13)}$$

Determining the constants from the above-mentioned asymptotic boundary conditions, we obtain:

$$\frac{d\varphi}{d\eta} = \frac{1}{1+\frac{a_1\eta^4}{\left(1+a_2\eta\sqrt{\eta}\right)^2}}, \text{ ------------------ (14)}$$

Where $a_1$=2,36172·10$^{-4}$, $a_2$=2,42979·10$^{-2}$.

Solving the last equation in quadratures:

$$\varphi = \int_0^\eta \frac{d\eta}{1+\frac{a_1\eta^4}{\left(1+a_2\eta\sqrt{\eta}\right)^2}}, \text{ --------------------------- (15)}$$

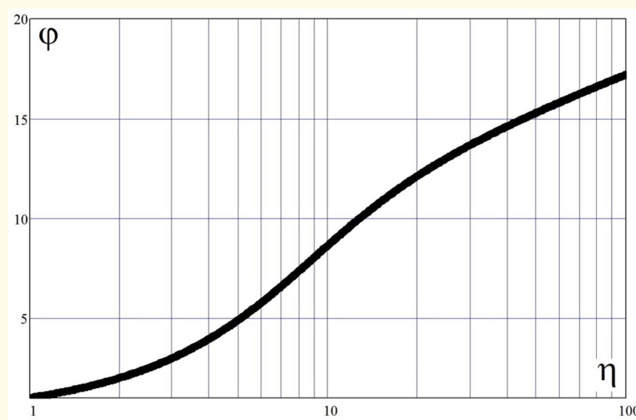


**Figure 1:** Theoretical profile of relative velocities from the thickness of boundary layers for turbulent flows based on numerical solutions of integral (15).

The numerical solution of this integral is shown in figure 1. It is not difficult to make sure that this solution correlates well with classical experimental data, e.g., in addition to the above: Laufer, Levich, Bauman Moscow State Technical University, Lin [1-3], etc.

For the last integral for small values of φ≤50, it is subsequently possible to obtain an approximate analytical approximation by taking:

$$\varphi = \int_0^\eta \frac{d\eta}{1+\frac{a_1\eta^4}{\left(1+a_2\eta\sqrt{\eta}\right)^2}} \cong \int_0^\eta \frac{d\eta}{a_3+a_4\eta^2}. \text{ ------------- (16)}$$

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Integral (16) can be analytically expressed:

$$\varphi \cong \int_0^{\eta} \frac{d\eta}{a_3 + a_4\,\eta^2} = a_5 \cdot \mathrm{arctg}(a_6\,\eta), \text{ -------------------(17)}$$

Where $a_5 = \frac{1}{\sqrt{a_4 a_3}} = 11{,}15;\ a_6 = \sqrt{\frac{a_4}{a_3}} = 0{,}097$

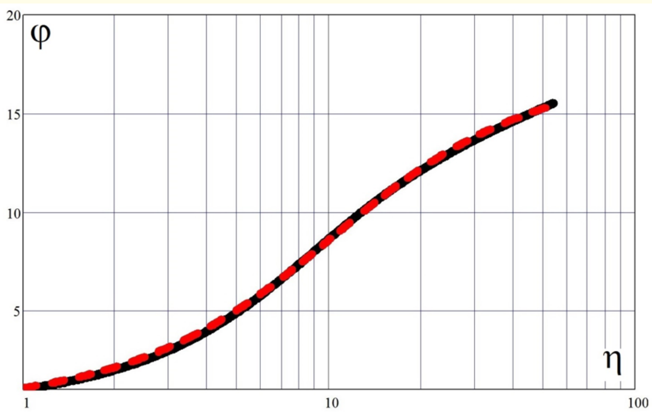


**Figure 2:** Comparison of the calculation results based on the approximate dependence (17) (dotted line) with the solution obtained by numerical integration (15) (continuous line) according to the theoretical profile of relative velocities along the thicknesses of boundary layers in turbulent flows.

A comparison of the calculated data according to the approximate formula (17) with the solution obtained by numerical integration (15), shown in figure 2, shows their almost complete identity in the approximated range().

The direct numerical solution of equation (2) is rather unstable. As an illustration, Figure 3 shows a comparison of the approximate solution (15) with the direct numerical solution of equation (2) when setting boundary conditions at the point n = 6.

As can be seen from figure 3, the correlation of the above solutions is very good, which indicates the adequacy of the proposed solutions by the method of sequential approximation.

Consequently, the approximate solutions obtained in this article, obtained by implementing the method of sequential approximation and using asymptotic functional boundary conditions, can be applied in practice, since they are in good agreement with both the experiment and the direct numerical solution of the basic differential equation.

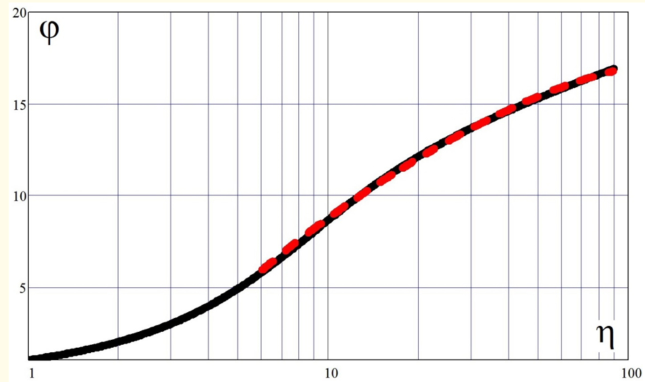


**Figure 3:** Comparison of the approximate solution (15) (continuous line) with the direct numerical solutions of the differential equation (2) (dotted line) for setting the boundary conditions at the point with η = 6.

## Conclusions

In the article, approximate analytical integral-approximation solutions were obtained using asymptotic functional boundary conditions using the sequential approximation method, which is in good agreement with both the direct numerical solution of the basic differential equation and classical experimental data.

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