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Research Article

# Study of Nonlinear Problem of Thermoelastic Beam Dynamics by Numerical Method

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## Abstract

The paper dwells on flexural vibrations of an elastic beam, the axial line of which in natural state is a plane curve and after loading remains plane. We made the following assumptions: all the cross-sections of beam remain plane and perpendicular to the longitudinal axis during deformation, the length of longitudinal axis is changing, the temperature field is stationary and nonhomogeneous. The thermoelastic curve for a beam is represented mathematically as functions of beam length and time: $v = v(l,t)$, $w = w(l,t)$, $\theta = \theta(l,t)$, where: $v, w$ and $\theta$ are respectively the vertical displacement, horizontal displacement and slope angle. To obtain these functions we have nonlinear differential equations, which contains the change in length of longitudinal axis.

The nonlinear differential equations of motion contain inertia of rotation of the cross-section. In particular, the paper describes the equations of small vibrations of a prismatic beam. To determine the range of natural frequencies of beam, the equations are written using boundary conditions. The impact of axial deformation and a temperature gradient on the frequencies is studied. Numerical calculations are carried out in a mathematical editor Mathcad.

**Keywords:** Thermoelastic Beam; Dynamics; Differential Equation; Frequency

## Introduction

Dynamic analysis of structures consists in determining internal forces and displacements from dynamic loads, or in testing the system for resonance with a recurrent load of certain frequency. The dynamics of structures is a lot more complicated than the statics of structures, and lags behind the latter in the development and improvement of applied calculation techniques. This is exemplified in numerical calculation applications for desktop software.

Therefore, a considerable amount of literature is available about the dynamic analysis of elastic beams. The monograph [1] explores the vibrations of elastic beams and problems characteristic of building structures in linear statement. In the works of V.A. Svetlitsky [2,3], nonlinear equations of the dynamics of elastic beams were obtained without considering deformation of axle, numerical methods for determining the natural frequencies and vibration modes were developed, and a number of applied problems of mechanical engineering and instrument-making were solved. The monograph [4] dwells on the computational models of movement of beam on the basis of the normal line hypothesis, as well as on the refined model in the light of the shift, and for determining the vibration frequencies, there are used the approximate analytical methods. The engineering calculation methods for problems of elastic beam dynamics are also addressed in the book [5].

Nonlinear free oscillations of initially direct Tymoshenko beam are investigated using asymptotic method [6]. Mechanical and geometric curvatures differ in the nonlinear setting of the problem and coincide in the linear problem (axial deformation $\varepsilon \ll 1$). The concept of "mechanical" curvature in the literature is used due to: simplicity, the ability to isolate the effect of pure bending from changes in curvature caused by tension. It has been shown that the two models provide the same nonlinear behavior for thin rods, while for thick rods there may be some difference up to 13%.

Article [8] presents a new equation in a closed form which accurately predicts the effect of an arbitrarily large constant axial load, residual stress or temperature change on the natural frequencies of a uniform single-span beam with different boundary conditions. Its accuracy and range of applicability are studied by comparing the results with numerical models and with approximate Bocaian formulas [7].

Article [9] deals with free bending vibration of the cantilever with an arbitrary axial load and a concentrated mass at the end. The effect of axial load and mass on the natural frequencies was analyzed, taking into account the inertia of the section rotation. The integral equation method was presented to solve the above problem. In particular, simple approximate expressions for natural frequencies were obtained for various cases. Considering the inertia of the rotation of the cross section reduces the natural frequencies [9]. The natural frequencies for the Euler-Bernoulli rod are overestimated. Tensile axial loading increases the natural frequencies, while compressive axial loading decreases them.

Article [10] describes transverse vibrations of symmetrical elastically connected thermoelastic system of two beams at uniform compressed axial load. The problem of mechanics is written as a nonlinear system consisting of hyperbolic and parabolic partial differential equations. The existence and singularity of the solution have been investigated.

The use of new light metal materials in building structures makes it relevant to study the influence of temperature factor on strength and rigidity within the elastic limits both under static and dynamic loads. The uneven temperature distribution, even at a small temperature change, can affect the free-running frequency. The use of numerical calculation methods allows for solving nonlinear problem, and it also automates the calculations for different fixing conditions and complex geometry of the axis of beam.

The purpose of the presented article is to develop methodology for calculating the dynamics of structural beam elements with allowance for an inhomogeneous temperature field.

## Methodology

### The equations of motion of beam in the plane.

Consider the vibration of beam in the plane, in which the centerline is located. We will investigate the movement of beam relative to the natural (unloaded) state. The temperature field is stationary and varies only in height of the cross section in the plane of a bend. In this formulation, the equations of statics at large displacements have been obtained in [11].

Consider the element of beam having the displacements in relation to the axes $y, z$ and the angular displacements relative to the axis $x$. In the initial state, the radius of curvature of the centerline we denote by $r_0$, and the slope angle of the tangent to the axis $z$ – by $\theta_0$. In general, the element of beam can be impacted by the distributed and concentrated forces and moments, which are the time-varying parameters.

Taking into account the forces and moments of inertia, the differential equations of motion for force factors take the form [12]:

$$\rho A(l)\frac{\partial^2 v}{\partial t^2} = \frac{\partial R}{\partial l} + q_y,$$
$$\rho A(l)\frac{\partial^2 w}{\partial t^2} = \frac{\partial H}{\partial l} + q_z, \qquad \text{-------------(1)}$$
$$\frac{\partial}{\partial t}(I_x^0\dot{\theta}) = \frac{\partial M}{\partial l} + m + R\cos\theta - H\sin\theta$$

Where: $\rho$ - the material density; $A$ - the cross-section area; $v, w$ - the displacements in the direction of the axes $y, z$, respectively; $I_x^0$ - the physical moment of inertia of the element of unit length beam. For the principal axes of the section $I_x^0 = \rho I_x$, $I_x$ - is a geometric moment of inertia of the cross section, $M$ - is a bending moment, $R, H$ - the components of the internal force vector, $q_y, q_z$ - the components of the vector of distributed external forces, and, $m$ - the intensity of the external bending moment.

For kinematics of deformation, the relations obtained in the work [11] are valid, replacing in them the ordinary derivatives by partial derivatives:

$$\frac{\partial v}{\partial l_0} = (1+\varepsilon_0)\sin\theta - \sin\theta_0$$
$$\frac{\partial w}{\partial l_0} = (1+\varepsilon_0)\cos\theta - \cos\theta_0 \qquad \text{---------- (2)}$$
$$\frac{\partial \theta}{\partial l_0} = \frac{1+\varepsilon_0}{r_0} + \kappa_x$$

Where: $\varepsilon_0$ - deformation of thermoelastic line, $l_0$ - arc length of undeformed thermoelastic line, $\kappa_x$ - characterizes variation of curvature.

The values $\varepsilon_0$ and $\kappa_x$ are determined by the formulas [11]:

$$\varepsilon_0 = \frac{N}{A^*} + \frac{1}{A^*}\int \varepsilon^T EdA,$$
$$\kappa_x = \frac{M}{I_x^*} + \frac{1}{I_x^*}\int \varepsilon^T yEdA \qquad \text{---------- (3)}$$

Where: $A^* = \int EdA$ - the generalized area; $I_x^* = \int y^2 EdA$ - the generalized moment of inertia; $E = E(T)$ - elastic modulus of material; $T = T(y)$ - temperature, $\varepsilon^T$ - temperature strain.

Normal force $N$ in the cross-section equals [11]:

$$N = H\cos\theta + R\sin\theta$$

According to the formulas (3), deformation ( $\varepsilon_0$ ) and curvature ( $\kappa_x$ ) are represented as the sum of the components from appropriate force factors and temperature:

$$\varepsilon_0 = \varepsilon_0^N + \varepsilon_0^T, \quad \kappa_x = \kappa_x^M + \kappa_x^T,$$
$$\varepsilon_0^N = \frac{N}{A^*}, \quad \kappa_x^M = \frac{M}{I_x^*}$$
$$\varepsilon_0^T = \frac{1}{A^*}\int \varepsilon^T EdA, \quad \kappa_x^T = \frac{1}{I_x^*}\int \varepsilon^T yEdA$$

For straight beams $r_0 \to \infty, \theta_0 = 0$ the equations (1) and (2) take the following form:

$$\rho A(z)\frac{\partial^2 v}{\partial t^2} = \frac{\partial R}{\partial z} + q_y,$$
$$\rho A(z)\frac{\partial^2 w}{\partial t^2} = \frac{\partial H}{\partial z} + q_z,$$
$$\rho I_x(z)\frac{\partial^2 \theta}{\partial t^2} = \frac{\partial M}{\partial z} + m + R\cos\theta, \qquad \text{------------ (4)}$$
$$\frac{\partial v}{\partial z} = \sin\theta,$$
$$\frac{\partial w}{\partial z} = \frac{\varepsilon_0}{1+\varepsilon_0}\cos\theta,$$
$$\frac{\partial \theta}{\partial z} = \frac{\kappa_x}{1+\varepsilon_0}$$

## Small free vibrations of rectilinear beam.

When designing the elastic beam elements operating in dynamic modes, it is necessary to determine the frequency range (more precisely, the several first frequencies), depending on the fixing conditions and the static stress-strain state. The frequencies are determined from the equations of small free vibrations of beam in relation to its natural state or relative to the equilibrium state.

We'll obtain the equations for small free vibrations of beam, assuming that the internal forces and displacements arising during vibrations are small. The components of the displacement and force vectors $(v, w, \theta, R, H, M)$ are first-order values of smallness, and because of this, we are neglecting their products in the derivation of the equations of motion. External loads are set equal to zero: $q_y = q_z = 0,$.

In the case of small displacements $\sin\theta \approx \theta,\ \cos\theta \approx 1, 1(1+\varepsilon_0) \approx 1-\varepsilon_0$. Consequently, the nonlinear equations (4) are simplified and take the form:

$$\rho A(z)\frac{\partial^2 v}{\partial t^2} = \frac{\partial R}{\partial z}, \qquad \frac{\partial v}{\partial z} = \theta,$$
$$\rho A(z)\frac{\partial^2 w}{\partial t^2} = \frac{\partial H}{\partial z}, \qquad \frac{\partial w}{\partial z} = \varepsilon_0,$$
$$\rho I_x(z)\frac{\partial^2 \theta}{\partial t^2} = \frac{\partial M}{\partial z} + R, \qquad \frac{\partial \theta}{\partial z} = \kappa_x. \qquad \text{---------- (5)}$$

Equations for an elastic beam without considering the deformation of axle and temperature changes are given in the book [2].

## Solution

To determine the frequencies of free vibrations, we represent the force and kinematic factors in the form as follows:

$$v(z,t) = v^*(z)e^{\omega t}, \qquad \kappa_x(z,t) = \kappa_x^*(z)e^{\omega t},$$
$$w(z,t) = w^*(z)e^{\omega t}, \qquad R(z,t) = R^*(z)e^{\omega t},$$
$$\theta(z,t) = \theta^*(z)e^{\omega t}, \qquad H(z,t) = H^*(z)e^{\omega t}, \qquad \text{--------(6)}$$
$$\varepsilon_0(z,t) = \varepsilon_0^*(z)e^{\omega t}, \qquad M(z,t) = M^*(z)e^{\omega t}$$

From the system of equations (5), with provision for the formulas (6), we obtain the ordinary differential equations:

$$\frac{dR^*}{dz} = -\rho A(z)\omega^2 v^*, \qquad \frac{dv^*}{dz} = \theta^*(z),$$
$$\frac{dH^*}{dz} = -\rho A(z)\omega^2 w^*, \qquad \frac{dw^*}{dz} = \varepsilon_0^*(z), \qquad \text{----------- (7)}$$
$$\frac{dM^*}{dz} = -\rho I_x \omega^2 \theta - R^*, \qquad \frac{d\theta^*}{dz} = \kappa_x^*$$

To integrate the system of differential equations (7), we need boundary conditions reflecting the end fixities of beam. For the numerical solution of problem, we introduce the dimensionless quantities:

$$\bar{v} = \frac{v^*}{L}, \quad \bar{w} = \frac{w^*}{L}, \quad \theta = \theta^*, \quad \bar{z} = \frac{z}{L},$$
$$\bar{\kappa}_x = \kappa_x^* L, \quad \bar{R} = \frac{R^* L^2}{E_0 I_{x0}}, \; \bar{H} = \frac{H^* L^2}{E_0 I_{x0}}, \; \bar{M} = \frac{M^* L}{E_0 I_{x0}}$$
$$\lambda_1 = \frac{A\rho}{A_0 \rho_0}, \quad \lambda_2 = \frac{\rho I_x}{\rho_0 A_0 L^2}, \quad \lambda_3 = \frac{E_0 I_{x0}}{L^2 A^*}$$
$$\lambda_4 = \frac{E_0 I_{x0}}{I_x^*}, \quad \bar{\varepsilon}_0^T = \frac{\varepsilon_0^T}{\alpha T_0}, \quad \bar{\kappa}_x^T = L\kappa_x^T$$
$$\Omega^2 = \rho_0 A_0 \omega^2 L^4 /( E_0 I_{x0})$$

Where: $T_0, E_0$ - is a room temperature and the corresponding modulus of elasticity of the material; $\alpha$ – coefficient of thermal expansion; $L$ - beam length, $A_0, I_{x0}$ - the area and moment of inertia of the cross-section at the origin of the coordinates.

In the dimensionless quantities, the equation system (7) takes the form as follows:

$$\frac{d\bar{R}}{d\bar{z}} = -\lambda_1 \Omega^2 \bar{v}, \qquad \frac{d\bar{v}}{d\bar{z}} = \theta,$$
$$\frac{d\bar{H}}{d\bar{z}} = -\lambda_1 \Omega^2 \bar{w}, \qquad \frac{d\bar{w}}{d\bar{z}} = \lambda_3 \bar{H} + \bar{\varepsilon}_0^T \qquad \text{------ (8)}$$
$$\frac{d\bar{M}}{d\bar{z}} = -\lambda_2 \Omega^2 \theta - \bar{R}, \qquad \frac{d\theta}{d\bar{z}} = \lambda_4 \bar{M} + \bar{\kappa}_x^T$$

The system of ordinary differential equations (8) can be written in vector-matrix form. For this purpose, let us introduce the following designations:

$$x_1 = \bar{v}, x_2 = \bar{w}, x_3 = \bar{\theta}, x_4 = \bar{R}, x_5 = \bar{H}, x_6 = \bar{M}$$

It follows that the equation system (8) will take the form as follows:

$$\frac{d\bar{X}}{d\bar{z}} = B\bar{X} + C \qquad \text{------- (10)}$$

Where:

$$\bar{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}, \; C = \begin{pmatrix} 0 \\ \bar{\varepsilon}_0^T \\ \bar{\kappa}_x^T \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_4 \\ -\lambda_1\Omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_1\Omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2\Omega^2 & -1 & 0 & 0 \end{pmatrix}$$

At both ends of the vibrating beam, there are six boundary conditions, from which it is possible to obtain the relations between the constants of general solution and frequency equation (a problem on eigenvalues). In this fashion, there will be determined the modes of free vibrations and their frequencies.

From the computational point of view, a problem on eigenvalues is very similar to the boundary problems, for which we use the shooting method. The difference lies in the adjustment not only by the missing left boundary conditions, but also by the target eigenvalues. To solve the problems on eigenvalues in a mathematical editor Mathcad, there are used the functions sbval.

## Results and Conclusion

Calculations were carried out for a free cantilever beam of the rectangular cross-section. The temperature in the section is changing by a square function. Boundary conditions take the form as follows: $x_1(0) = 0;\ x_2(0) = 0;\ x_3(0) = 0;\ x_4(1) = 0;\ x_5(1) = 0;\ x_6(1) = 0$. Figure 1 illustrates the calculation results.

Designations: s1 $\equiv \bar{z}$, s2$\equiv \bar{v}$, s3$\equiv \bar{w}$,

s4$\equiv \theta$, s5$\equiv \bar{R}$, s6$\equiv \bar{H}$, s7$\equiv \bar{M}$.

For strength analysis, of high significance is the first free-running frequency. By calculation, it equals to 6,2 and is about 11% less than for elastic beam.

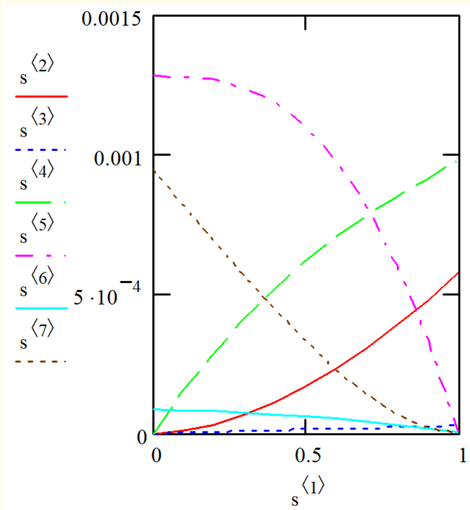


**Figure 1:** The calculation results.

Results of numerical calculation of dependence of natural frequency on temperature gradient are given in work [13].

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