Acta Scientific COMPUTER SCIENCES

Volume 3 Issue 11 November 2021

Research Article

On Bivariate Copula Modelling: An application to Infant Mortality and Fertility Rate Data

Gabriel Asare Okyere*, Godfred Zaachi, Emmanuel K Owusu Mintah, Charles K Amponsah and Daniel A Alhassan

Department of Statistic and Actuarial Science, Kwame Nkrumah University of Science and Technology, PMB, Kumasi, Ghana

***Corresponding Author:** Gabriel Asare Okyere, Department of Statistic and Actuarial Science, Kwame Nkrumah University of Science and Technology, PMB, Kumasi, Ghana.

Received: June 21,2021

Published: October 27, 2021

Abstract

Infant survival is key to the new era of Sustainable Development Goals (SDGs). However, in Africa and Ghana especially, this is challenged by the alarming Infant Mortality Rate (IMR) - a direct consequence of population growth characterized by high fertility. The purpose of our study is to investigate the extent to which Infant Mortality Rate(IMR) occur conditionally on Total Fertility Rate(TFR) over the years and how these indicators are related using data from 1960 to 2020. We highlight the application of copula models in dealing with interdependencies between IMR and the TFR. In this study we compare several copula models using the differences in Akaike Information Criterion (AIC) to select the most appropriate" model for our data. The results indicate that the bivariate Clayton copula with continuous. Weibull margins best" describe the conditional distribution of IMR. Our results further indicate a 90.40% chance that IMR will exceed 120.5 deaths per 1000 live births if the TFR rise to 7 children per woman. We also conclude base on our model estimate that 2021 infant mortality will exceed its 2020 estimated value of 32.80 deaths per 1000 live births given the current fertility rate of 4 births per woman with a chance as low as 2%.

Keywords: Aikaike Information Criteria (AIC); Copula; Infants Mortality Rate; Marginal Distribution; Total Fertility Rate

Introduction

The transition of fertility and mortality levels from high to low is a key characteristic of a developmental process in any demographic region. The timing of fertility and mortality have varied considerably over the years. According to the general child survival hypothesis, a lower growth in the population is a consequence of reduced child mortality that is followed by a reduced fertility [29]. According to Centers for Disease Control and Prevention (CDC) USA, infant mortality is the death of an infant before his or her first birthday and hence the infant mortality rate (IMR) is the number of infant deaths for every 1,000 live births. The common causes range from infection to birth defects or accidents. However, the main causes of infant mortality is different and more pronounced in developing countries in Africa than it is in developed countries. In 2017 alone, 4.1 million deaths occurred in the first year of life [24]. The risk of a child dying before completing the first birthday of life was 51 per 1000 live births in Africa compared to 8 per 1000 live births in Europe, which is over six times higher. Despite the global reduction in infant mortality rate from 65 deaths per 1000 live births in 1990 to 29 deaths per 1000 live births in 2017, maternal and child health still poses a threat [27].

Total Fertility Rate (TFR) on the other hand defines the average number of children a woman would have assuming the current age specific birth rates remain constant throughout her childbearing years (15-49 years) [8]. It is calculated by summing across the average number of births per woman in five-year age groups. That is

$$TFR = 5 \times \sum_{k=1}^{7}(ASFR_k) = 5 \times \left(\frac{N_{bw}[15-19]}{P_w[15-19]} + \cdots + \frac{N_{bw}[45-49]}{P_w[45-49]}\right) \text{----(1)}$$

Where, N_{bw}: number of births to women aged, P_w: population women aged, k: five-year age groups, $ASFR_k$: Age specific fertility rate in the kth age group. This means that TFR depends not only on the number of births but also on the number of women across the childbearing age groups.

While fertility rates are falling globally, it is important to note that an increase in the number of births does not necessarily lead to an increase in TFR. In 1950, women had an average of 4.7 children in their childbearing years. However, by 2017, the global fertility rate nearly halved to 2.4 and it is estimated to fall below 1.7 by 2100 [20]. In Ghana, the TFR was 3.93 children per woman in 2017 compared to 4.27 children per woman in 2010. Although there is a fall in fertility rate, the women resident in the rural areas in Ghana contributed more having a higher fertility rate of 5.44 compared to those in the urban area (fertility rate 3.92) [9]. The global fall in TFR has been attributed to more women in education and greater access to contraception [10] both being characteristics of urban life in Ghana. Yet, IMR declined from 64 deaths per 1000 live births in 2003 to 50 deaths per 1000 live births in 2008 (GDHS, 2008) and from 47.5 deaths per 1000 live births in 2010 to 32.8 deaths per 1000 live births in 2020 representing a 41% drop in mortality over a decade indicating an increasing population trend.

Past studies have shown that there is a relationship between a nation's IMR and her TFR. [17] discuss the effects of IMR and TFR in Turkey [3]. Recounts, a regression model that incorporates infant mortality as one of the predictors of TFR with the study indicating a positive relationship between infant mortality ratio and TFR. [15] examined the relationship between female labor force participation rates, infant mortality rates and fertility rates in Australia using Granger Causality tests. Modeling via copulas provides much flexibility as it allows the researchers to specify models for marginal distributions separately from the dependence structure that links the models to form a joint distribution [18]. This frees the researcher to consider a much wider class of multivariate distributions other than the commonly existing ones in the literature. Copula-based models have gained much popularity and success in the econometric and finance fields. In this paper, we apply copula-based approach to model the distribution between the TFR and IMR, while taking into account the dependencies between them. We also compare several copula models using differences in the Akaike Information Criterion (AIC) to select the "most appropriate" model for our data.

Our work is novel in two ways: (1) we use a copula model to study and capture the scale-free interdependencies [6] between the TFR and IMR and (2) describe the extent to which IMR occur conditionally on the TFR (conditional survival rate).

The rest of this paper is organized as follows. Section 4 describes the methods involved. We discuss the marginal distribution criteria, some bivariate copula models and the estimation procedure used. The results and discussions are presented in Section 5 and Section 6 concludes with a discussion of future research.

Materials and Methods

In this section we present the data source and the methods used for our study. Our copula analysis is based on the elliptical family (Gaussian and t copulae) and the Archimedean family (Clayton, Frank, Joe and Gumbel copulae). We also consider the following two-parameter mixed distributions: Clayton-Gumbel (BB1), Joe-Gumbel (BB6), Joe-Clayton (BB7) in addition to Tawn 1 and Tawn 2. The distributional properties of these copulae are stated in appendix A.

Data and source

The Data used for this study contains IMR and TFR indicators of Ghana from 1960 to 2020 obtained on-line from the Ghana Economic outlook and world-bank as referenced [5,8,16,17,24].

Copulas and sklar's theorem

Informally, Copulas are functions that "couple" multivariate distribution functions to their marginal distributions [20]. According to Sklar [22], an n-dimensional joint distribution can be decomposed into its n-univariate marginal distributions and an n- dimensional copula. For the purposes of our work, we take n = 2. To this end, let $Y = (Y_1, Y2)^{\mathrm{I}}$ be a random vector with cumulative distribution function G and, for $i \in \{1, 2\}$, let F_i denote the marginal

distribution of Y_i. Then there exists a copula C: $[0, 1]^2 \rightarrow [0, 1]$ such that, for all $y = (y_1, y_2) \in R^2$,

$$G(y) = C\{F_1(y_1), F_2(y_2)\} \text{-------(2)}$$

Definition (Copula)

A bivariate copula C, of the random vector Y is thus a function that maps the univariate marginal distributions F_1, F_2 to the joint distribution G, and we write $Y \sim G = C(F_1, F_2)$. If F_1, F_2 continuous, then C is unique, which is the case we consider here. The usefulness of (2) is seen in the converse of Sklar's theorem: if C is a copula and F_1 and F_2 are distribution functions, then the function G defined by (2) is a joint distribution function with marginals F_1 and F_2.

$$F(x,y) = C[F_1(x), F_2(y)],\ x, y \in R, \text{---(3)}$$

Modeling with copula

As a direct consequence of equation (3), a model for (X, Y) can be structured with $C \in C(\theta)$, $F_1 \in F_1(\sigma)$, $F_2 \in F_2(\gamma)$ selected from a known parametric family with parametric vectors, θ, σ and γ. The joint density of the bi-variate cdf with uniform marginals is bounded by the so-called Frechet - Hoeffding bound defined as

$$max\left[\sum_{i=1}^{2} U_i - 1, 0\right] \leq (F(x), F(y)) \leq min(F(x), F(y)) \text{-------(4)}$$

Where U_i are uniform random variates. More generally, the copula representation for the Frechet - Hoeffding bounds is defined as $W(u_1,u_2) \leq C(u_1,u_2) \leq M(u_1,u_2)$, $u_i \in [0,1]$ see [13].

Selecting marginal distributions

Model selection criteria based on Akaike Information Criterion is defined in terms of copula density, c as;

$$AIC = -2\sum_{i=1}^{n} \ln[c(u_{i1}, u_{i2} \mid \theta)] + 2K. \text{---------(5)}$$

Where, N, K are the number of data points and model parameters respectively. To allow for quick comparison and ranking of candidate models, the AIC difference, $\Delta AIC_j = AIC_j - \min(AIC)$, is used herein. See [1].

Fitting copula models

In this section, the copula analysis is based on the Elliptical family (Gaussian and t-copula) and Archimedean family (Clayton, Frank, Joe, Gumbel). Other two parameter mixed distribution as Clayton - Gumbel (BB1), Joe- Gumbel (BB6), Joe - Clayton (BB7) as well as Tawn 1 and Tawn 2 are herein considered. The distributional properties of these copulae are attached as appendix A.

Kendall's tau and Tail dependence

The Kendall's tau in a bivariate copula C with uniform marginals is given as;

$$\tau_k(X,Y) = 4\int_0^1\int_0^1 C(u,v)dC(u,v) - 1 = E[C(UV)] - 1 \text{-------(6)}$$

Where E[C (U, V)] is the expected value of the joint copula distribution. See [14]. The index of upper tail dependence, $I_U = \lim_{u\to 1} \frac{\overline{C(u,u)}}{1-u}$ is an upper tail dependence if $I_U \in (0, 1]$ and no dependence in the upper tail if $I_U = 0$. Also, if $I_L = \lim_{u\to 0} \frac{C(u,u)}{u}$ exist, then C has dependence in the lower tail if $I_L \in (0, 1]$ and no dependence in the lower tail if $I_L = 0$.

Maximum Likelihood Estimation (MLE)

Let $(y_1, \cdots, y_n)$ be observations with $f_j(.;\theta_j)$ and $F_j(.;\theta_j)$ as the j^{th} marginal density and distribution function respectively. Then the MLE in the bivariate case involves maximizing the log likelihood.

$$l(\alpha,\beta_{\ }) = \sum_{i=1}^{N} \log c\left[F_1(y_{1i} \mid x_{1i},\beta_{\ 1}), \cdots F_2(y_{2i} \mid x_{2i},\beta_{\ 2}); \alpha\right] + \sum_{i=1}^{N}\sum_{j=1}^{2} \log f_i(y_{ij},\beta_{\ j})\Big), \quad (7)$$

Where $c\ F_1(.)F_2(.)) = \frac{d}{dxdy} C\ F_1(.)F_2(.))C_{12}\ F_1(.)F_2(.))f_1(.)f_2(.)$

see [24]. The ML estimator of θ is $\hat{\theta} = \text{argmax}_{\hat{\theta}} l(\theta), \theta \in \Theta$, where α denote the parameter of the copula C and θ is a parameter vector. The parameter estimation is based on the two - staged Inference Functions for Marginals (IFM) method (due to [23]) stated without loss of generality as;

$$\hat{\alpha}_{IFM} = \text{argmax}_{\hat{\alpha}} \sum_{i=1}^{N} \log c\left(F_1\left(x_{i,1};\hat{\beta}_{IFM}\right), \ldots, F_i\left(x_{i,d};\hat{\beta}_{IFM}\right), \alpha\right) - - - - - \ (8)$$

and

$$\hat{\beta}_{IFM} = \text{argmax}_{\hat{\alpha}} \sum_{i=1}^{n} \log f_i\left(x_{ij};\beta_j\right) - - - - - \ (9)$$

Results and Discussion

Preliminary analysis

Summary statistics

Finding drawn from summary statistics for the entire 60 year period reveal that the distributions of IMR and TFR are less peaked

than normal with respective estimated kurtosis of 1.59 and 1.43 against 3.00 (normal kurtosis). The empirical mean for infant mortality is estimated to be 82.01 deaths per 1000 live births with corresponding fertility rate of 5.58 births per woman in her active reproductive life cycle.

Correlation analysis

Figure 1 below shows a pairwise correlation obtain via the Kendall's tau, τ_k correlation measure along with bivariate scatter plot and probability histogram on IMR and TFR indicators in Ghana. The correlation coefficient show a strong association measure of 0.92 between IMR and TFR.

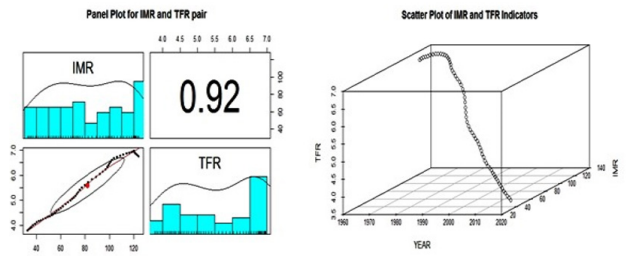

Figure 1: Correlation analysis on IMR and TFR in Ghana from 1960 - 2020.

Choice of a copula model

Assuming our data, consisting of IMR and TFR follows the various copula models (Elliptical family, Archimedean family, and others), we estimated their parameters via MLE, the Kendall tau τ_k, tail dependency, and the Akaike Information Criterion (AIC) for each copula. Table 1 below shows the estimated parameters from fitted copulae with their respective standard errors in the bracket, τ_k, lower and upper tail dependence, and Akaike information criteria. While it is important to note that a tail dependence measure of zero ($I_i = 0$) as in the case of Frank and Gaussian copulae does not suggest independence, only BB1 accounted for dependence in the upper and lower tails of its distribution. BB7 copula has a lower tail dependence measure of 0.95 whereas the Tawn 1 has the highest upper tail dependence of 0.85 with Gumbel, Tawn 2, and BB6 having the same tail measure of 0.80. Generally, the Clayton copula is the best in terms of fit with ΔAIC = 0.00 and a lower tail dependence measure of 0.98; thus, there is a 98 percent probability that IMR falls below a certain rate condition that the TFR also falls below a certain rate in the same time period.

Copula	Parameter Estimate	τ_k	IL	IU	AIC	Δ AIC
Clayton	α = 32.62 (4.02)	0.94	0.98	0.00	-284.80	0.00
Gumbel	α = 3.87 (0.46)	0.74	0.00	0.80	-95.01	189.79
Frank	α = 37.27(4.68)	0.90	0.00	0.00	-187.40	97.40
Joe	α = 3.52 (0.48)	0.57	0.00	0.78	-56.51	228.29
Gaussian	ρ = 0.94 (0.01)	0.77	0.00	0.00	-118.46	166.34
student -t	par1 = 0.99 par2 = 2.00	0.90	-	-	-179.25	105.55
Tawn 1	par1 = 5.93(NA) par2 = 0.94 (NA)	0.79	0.00	0.85	-107.46	177.34
Tawn 2	par1 =3.93(NA) par2 = 0.99 (NA)	0.74	0.00	0.80	-92.61	192.19
BB1	θ = 5.00 (NA) δ = 2.88 (NA	0.90	0.95	0.73	-219.41	65.39
BB6	θ = 1.00 (NA) δ = 3.86 (NA)	0.74	0.00	0.80	-92.97	191.83
BB7	θ = 1.00 (NA) δ = 6.00 (NA)	0.75	0.89	0.00	-179.78	105.02

Table 1: Results for fitting bivariate Copula to IMR and TFR data.

Choice of marginal distribution

We chose the marginal distributions of IMR and TFR base on the smallest value of ΔAIC. A summary of results from fitted marginals which includes the parameters estimates with standard errors in brackets, AIC, and ΔAIC are shown in table 2 below. The empirical result based on ΔAIC from Table 2 shows that the Weibull marginal is ideal for IMR and TFR in Ghana.

Results based on ΔAIC from table 2 above, suggest the Weibull marginal as ideal for IMR and TFR data in Ghana.

	IMR			TFR		
Distribution	**Parameter Estimate**	**AIC**	**Δ AIC**	**Parameter Estimate**	**AIC**	**Δ AIC**
Exponential	Rate = 0.012 (0.002)	661.64	72.41	Rate = 0.18 (0.02)	333.79	145.50
Gamma	Shape = 6.62 (1.17) Rate =0.04(0.01)	593.09	3.86	Shape = 24.01 (4.32) Rate = 4.30(0.78)	191.32	3.03
Lognormal	Meanlog= 4.33(0.05) Sdlog = 0.41 (0.04)	596.47	7.24	Meanlog =1.70(0.03) Sdlog = 0.21 (0.02)	192.37	4.08
Weibull	Shape = 3.10 (0.33) Scale = 92.09(4.00)	589.23	0.00	Shape = 5.94(0.63) Scale = 6.04 (0.14)	188.29	0.00
Normal	Mean = 82.01 (3.84) Sd = 29.98 (2.71)	591.99	2.76	Mean = 5.58 (0.14) Sd = 1.12(0.10)	190.49	2.20
Pareto	Shape = 0.47 Scale = 39.61	663.64	74.41	Shape = 0.47 Scale = 2.68	321.35	133.06
Loglogistic	Shape = 4.07 (0.42) Scale = 78.17 (4.39)	601.14	11.91	Shape = 7.79 (0.79) Scale = 5.52 (0.16)	199.00	10.71
Student- t	df = 0.19 (0.02)	951.44	362.21	df = 0.49 (0.08)	537.81	349.52

Table 2: Summary results from Fitting IMR and TFR in Ghana from 1960 - 2020.

Fitting copula with continuous marginals

The panel plots compares results from 5000 simulated samples from Clayton, Frank and Gaussian copulae with bivariate Weibull marginals.

The τ_k from the simulated Clayton copula with bivariate Weibull marginals is 2% stronger than the empirical τ_k (0.92) when compared. In addition, fitted results as per figure 2 are shown in table 3 below. Where Margin 1: Marginal for IMR, Margin 2: Marginal for TFR.

Further analysis with clayton copula

The suitability of the resulting Clayton copula model is compared with the fit from 5000 simulated samples from arbitrary marginals corresponding to IMR and TFR indicators as shown in table 2. The results along with their ΔAIC values are shown in table 4 below.

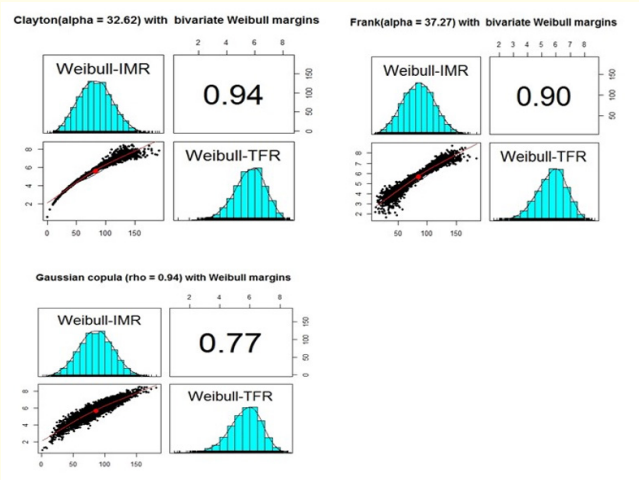

Figure 2: Panel Plot using Clayton, Frank and Gaussian copulae with Weibull margins.

Bivariate Copula	Copula	Margin 1	Margin 2	AIC	Δ AIC
Clayton	α = 31.76(0.59)	Shape = 3.14(0.03) Scale = 92.87(4.00)	Shape = 6.01(0.06) Scale = 6.07(0.01)	37452.00	00.00
Frank	α = 36.02(0.46)	Shape = 3.52(0.03) Scale = 95.16 (0.30)	Shape = 6.37(0.06) Scale = 6.12 (0.01)	44856.00	7404.00
Gaussian	ρ = 0.94(0.00)	Shape = 3.52 (0.04) scale = 94.97 (0.4)	Shape = 6.35 (0.07) Scale = 6.12 (0.01)	50974.00	13522.00

Table 3: Summary of fitted bivariate copula.

Copula	Margin 1	Margin 2	AIC	Δ AIC
Clayton	Weibull	Weibull	37452	0000.00
Clayton	Lognormal	Lognormal	38648	1196.00
Clayton	Weibull	Lognormal	39596	2144.00
Clayton	Lognormal	Weibull	39446	1994.00
Clayton	Exponential	Weibull	46452	9000.00
Clayton	Weibull	Gamma	39310	1858.00
Clayton	Weibull	Normal	39296	1844.00
Clayton	Loglogistic	Gamma	41228	3776.00

Table 4: Summary of fitted Clayton Copula with arbitrary marginals.

From table 4 above, it is clear that the Clayton copula better describes the conditional distribution of IMR given TFR. The cumulative and survival densities of the bivariate Clayton copula with α = 31.76 are provided as Appendix A. The results show there is 27% chance that infant mortality will exceed 124.30 given that total fertility is 6.75 births per woman. Thus, if IMR falls below its 1960 estimate of 124.30 deaths per 1000 live births, there was approximately 73% chance that TFR will rise above 6.75 births per woman in the following year (1961). Similarly, there was only a 3% chance that in 2020 Infant mortality rate (32.80) will exceed its 2019 estimate of 33.80 deaths per 1000 live births given the total fertility rate was 3.82 births per woman. It is projected that by the end of 2021, there is only 2% chance that infant mortality will exceed its 2020 estimated value of 32.80 deaths per 1000 live births given the current fertility rate of 4 births per woman.

Conclusion

The Clayton copula with bivariate Weibull marginals provide an ideal description for the conditional distribution of IMR given TFR in Ghana. The estimated parameter value (α) for the bivariate Clayton copula is 31.76 (high) suggesting a strong dependence measure of 0.94 in the lower tail. In addition, the probability densities revealed that there can be as high as 90.40% chance that in a given year the IMR can exceed 120.50 deaths per 1000 live births given that the TFR in the country is 7 children per woman. This is about 42.00% higher than the average IMR of 85.04 deaths per 1000 live births for the entire period.

Similar trends inherent from the probability estimates in Appendix A attest that population growth has a direct consequence on IMR. Also, the conditional density estimates project that by the end of 2021, there could be as low as 2% chance that infant mortality will exceed its 2020 estimated value of 32.80 deaths per 1000 live births given the current fertility rate of 4 births per woman.

Conflict of Interest

We declare that no financial interest or any conflict of interest exists in the conduct of this research work.

	IMR	TFR	C(u,v)		IMR	TFR	C(u,v)
[1,]	124.30	6.75	0.73	[31,]	79.80	5.60	0.49
[2,]	123.10	6.79	0.76	[32,]	77.50	5.50	0.47
[3,]	122.10	6.83	0.79	[33,]	75.60	5.40	0.46
[4,]	121.40	6.86	0.81	[34,]	74.30	5.32	0.44
[5,]	121.00	6.89	0.85	[35,]	73.30	5.24	0.43
[6,]	120.80	6.92	0.87	[36,]	72.30	5.17	0.41
[7,]	120.70	6.94	0.87	[37,]	71.10	5.10	0.39
[8,]	120.70	6.95	0.87	[38,]	69.70	5.04	0.38
[9,]	120.70	6.96	0.87	[39,]	68.10	4.97	0.36
[10,]	120.50	6.96	0.84	[40,]	66.20	4.90	0.35
[11,]	120.10	6.95	0.82	[41,]	64.20	4.83	0.33
[12,]	119.30	6.94	0.81	[42,]	62.10	4.75	0.32
[13,]	118.10	6.92	0.79	[43,]	60.20	4.68	0.30
[14,]	116.40	6.89	0.77	[44,]	58.50	4.61	0.28
[15,]	114.20	6.86	0.76	[45,]	56.90	4.55	0.27
[16,]	111.60	6.82	0.74	[46,]	55.50	4.49	0.25
[17,]	108.90	6.77	0.72	[47,]	54.20	4.44	0.24
[18,]	106.20	6.72	0.69	[48,]	52.80	4.39	0.22
[19,]	104.00	6.67	0.68	[49,]	51.20	4.35	0.21
[20,]	102.30	6.61	0.66	[50,]	49.40	4.31	0.19
[21,]	101.10	6.54	0.65	[51,]	47.50	4.27	0.17
[22,]	100.20	6.47	0.63	[52,]	45.60	4.23	0.16
[23,]	99.50	6.39	0.62	[53,]	43.70	4.19	0.14
[24,]	98.40	6.31	0.60	[54,]	42.00	4.15	0.13
[25,]	96.90	6.22	0.58	[55,]	40.40	4.10	0.11
[26,]	94.70	6.13	0.57	[56,]	38.90	4.04	0.09
[27,]	92.10	6.03	0.55	[57,]	37.40	3.98	0.08
[28,]	89.00	5.93	0.54	[58,]	36.10	3.93	0.06
[29,]	85.80	5.82	0.52	[59,]	34.90	3.87	0.05
[30,]	82.70	5.71	0.51	[60,]	33.88	3.82	0.03
				[61,]	32.80	3.77	0.02

Appendix A: C(u ,v), u, v [0,1].

Note: : $\overline{C(u,v)} = 1 - C(u,v)$

Family	$C(u_1, u_2)$	Generator, $\varphi(t)$	I_L	I_U	τ_k	Parameter
Elliptical						
Gaussian	$\Phi_G[\Phi^{-1}(u_1), \Phi^{-1}(u_2); \theta]$		-	-	$\frac{2}{\pi}\arcsin(\rho)$	
Student -t	$\int_{-\infty}^{t_v^{-1}(u_1)} \int_{\infty}^{t_v^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} exp\left(1 + \frac{x^2+y^2-2\rho xy}{v(1-\rho^2)}\right) dy\, dx^{-(1+v/2)}$		0	$2t_{v+1}\left(-\sqrt{\frac{1-\rho}{1+\rho}}(v+1)\right)$	$\frac{2}{\pi}\arcsin(\rho)$	
Archimedean						
Independent	$u_1 u_2$	$-\ln t$	-	-	0	NA
clayton	$(u_1^{-\alpha} + u_2^{-\alpha} - 1)^{-1/\alpha}$	$t^{-\alpha} - 1$	$2^{-1/\alpha}$	-	$\alpha/(\alpha+2)$	$\alpha > 1$
Frank	$\frac{1}{\alpha} ln\left(1 + \frac{(e^{\alpha u_1}-1)(e^{\alpha u_2}-1)}{e^{\alpha}-1}\right)$	$ln\left(\frac{e^{\alpha t}-1}{e^{\alpha}-1}\right)$	0	0	$1 - \frac{4}{\alpha}\{D(-\alpha) - 1\}$	$-\infty < \alpha < \infty$
Gumbel	$exp - [(-\ln u_1)^{\alpha} + (-\ln u_2)^{\alpha}]^{1/\alpha}$	$(-\ln t)^{\alpha}$	0	$2 - 2^{1/\alpha}$	$1 - 1/\alpha$	$\alpha \geq 1$
Joe	$[(1-u_1) + (1-u_2) - (1-u_1)^{\alpha}(1-u_2)^{\alpha}]^{1/\alpha}$	$-\ln 1 - (1-t)^{\alpha}$	0	$2 - 2^{1/\alpha}$	$1 + \frac{4}{\alpha}\int_0^1 y\ ln\, y(1-y)^{2(1-\alpha)\alpha}\, dy$	$\alpha \geq 1$
BB1	$\left[1 + \sum_{d=1}^{2}(u_d^{-\delta} - 1)^{\frac{1}{\theta}}\right]^{-\frac{1}{\delta}}$	$(t^{-\delta} - 1)^{\theta}$	$2^{-\frac{1}{\delta\theta}}$	$2 - 2^{1/\theta}$		$\delta > 0, \theta \geq 1$
BB6	$1 - \left\{1 - exp\left[-\left(ln \sum_{d=1}^{2}\{-ln[1-(1-u_d)^{\theta}]\}^{\delta}\right)\right]^{\frac{1}{\delta}}\right\}^{\frac{1}{\theta}}$	$\{-ln(1-(1-t)^{\theta}$	0	$2 - 2^{1/\delta\theta}$		$\theta \geq 1, \delta > 1$
Where, $t_v^{-1}(.)$: quantile function of the student-t with v degrees of freedom, D(.) is Debye function $\frac{k}{x^k}\int_0^x \frac{t^k}{e^t - 1}\, dt$						

Appendix B: Some Copula families.

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