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Review Article

Underwater Acoustic Localization Using Nonlinear Processing

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Abstract

The synthesis of coupled systems including linear propagation media and nonlinear lumped subsystems is investigated. The resulting coupled system is expected to exhibit improved dynamic behavior. Such improvements are sought after by designing exclusively the lumped nonlinear subsystem and not by modifying the propagation medium. The lumped subsystem can be static or dynamic as well as passive or active. The design method is based on the Volterra-Wiener theory of nonlinear systems combined with the Linear Fractional Transformation employed for the analysis of uncertain linear systems. Although the techniques are applicable to a variety of practical engineering and physical systems, this work addresses acoustic source localization. Indeed, a moving acoustic source can be considered to nonlinearly modify the characteristics of a carrier acoustic wave. Such an acoustic carrier might be a monochromatic or polychromatic tone as well as other broadband signals such as band-limited white or colored noise. The sound source position signal is propagated through a nonlinear operator and then, under noise-free environment assumptions, determines the sound pressure level at the receiver location. In this paper, the proposed method is applied to the simplified case of a one-dimensional acoustic cavity defined by totally reflective barriers. Furthermore, the lossless wave equation is assumed to govern the sound pressure level in the homogeneous domain of interest. However, even in this simple scenario, only the additional assumptions of negligible source velocity and acceleration allow the derivation of an expression for the sound pressure level containing exclusively source displacement. In this context, simulation data series or analytical solutions, approximate or exact, are post-processed in order to determine the Volterra kernels, which effectively are a convenient extension of the impulse response concept in the nonlinear case, of the operator connecting source displacement to sound pressure level at the receiver. The outcome is Linear Time Invariant models depicting the dominant sound propagation dynamics. Then the synthesis stage is based on the properties of interconnected nonlinear systems that are described in the Volterra-Wiener framework. By using such properties, the signal processing algorithm for the estimation of the acoustic source position based on the received sound pressure level is finally developed.

Keywords: Linear Systems; Frequency; Acoustic Pressure

Introduction

An improved source localization technique can be achieved by employing tools for the analysis of coupled systems with linear propagation mediums and nonlinear lumped subsystems. In effect, a lumped nonlinear signal analysis system is proposed that can provide ocean acoustic source localization on the basis of some prior data and knowledge about the propagation medium. The design method will be based on the Volterra-Wiener theory of nonlinear systems combined with the Linear Fractional Transformation employed for the analysis of uncertain linear systems. Although the techniques are applicable to a variety of practical engineering and physical systems this work will focus on ocean acoustic source localization.

Synthesis of feedback systems is a process relying greatly on modeling of the systems under investigation. In the case at hand, the linear or linearized systems under consideration can conventionally be described based on the theoretical framework of partial differential equations in the frequency domain. In this respect, the analysis here includes the detailed modeling of distributed parameter dynamic systems. Using such models, a formal reduction procedure is applied in order to formulate a finite-order problem of coupling of the linear, distributed system with a nonlinear, lumped element which can be either passive or active and either dynamic or static. The application range of the method proposed here, although broad by nature, will be focused on acoustics and in specific ocean acoustics. Other applications include ocean wave energy systems like e.g. the Oscillating Water Column (OWC) and mechatronic systems like e.g. MEMS.

Problem formulation

At first, numerical modeling combined to model order reduction for distributed linear systems needs be employed. In the case of underwater acoustic wave propagation, the problem can be formulated as the analysis of the development and propagation of the acoustic field in confined spaces, like e.g. the one shown in Figure 1 for the 1D case. If acoustic waves are generated by a point source located at point at time instant t, the propagation is governed by the lossless wave equation for the sound pressure in the homogeneous domain of interest.

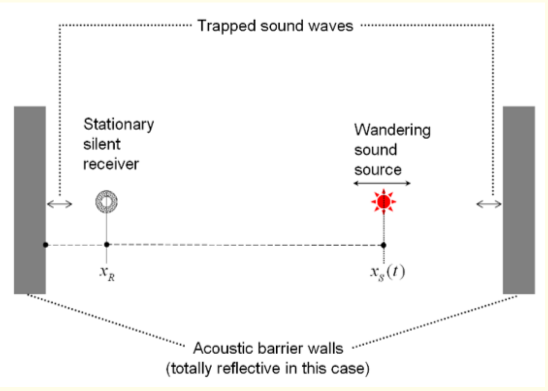

Figure 1: Problem setup.

$$\frac{\partial^2 p}{\partial t^2} = c^2\nabla^2 p + m(t)\cdot\delta\left(x - x_S(t)\right) \tag{1}$$

In the above, p stands for the acoustic pressure, ω_s is the excitation frequency and c the sound propagation speed in the acoustic medium. Finally, m(t) is the acoustic signal emitted by the wandering source. For the rest of this analysis sinusoidal time dependence will be assumed.

$$m(t) = \exp\left(j\omega_S t\right) \tag{2}$$

Sound pressure dependence on source motion

The analysis starts by considering the homogeneous wave equation, i.e. the one with the source silent.

$$\frac{\partial^2 p}{\partial t^2} = c^2\nabla^2 p \Rightarrow \frac{\partial^2 p}{\partial t^2} = c^2\frac{\partial^2 p}{\partial x^2} \tag{3}$$

This needs to be solved in the bounded domain with fully reflective boundary conditions. Any function of the following form is a solution of the homogeneous wave equation in unbounded domains.

$$p(x,t) = p_+(ct - x) + p_-(ct + x) \tag{4}$$

By employing the Fourier transform:

$$\begin{gathered} p_\pm(z) = \int_{-\infty}^{+\infty} P_\pm(k)\exp(j2\pi kz)\,dk, z = ct \mp x \\ \Downarrow \\ p_\pm(z) = \int_{-\infty}^{+\infty} P_\pm(k)e^{j\omega t}e^{\mp j2\pi kx}\,dk, \omega = c\cdot 2\pi k \end{gathered} \tag{5}$$

By substituting the above in the homogeneous wave equation one obtains the following.

$$\begin{gathered} \frac{\partial^2}{\partial t^2}\int_{-\infty}^{+\infty} e^{j\omega t}\left[P_+(k)e^{-j2\pi kx} + P_-(k)e^{j2\pi kx}\right]dk = \\ = c^2\frac{\partial^2}{\partial x^2}\int_{-\infty}^{+\infty} e^{j\omega t}\left[P_+(k)e^{-j2\pi kx} + P_-(k)e^{j2\pi kx}\right]dk \\ \Downarrow \\ \int_{-\infty}^{+\infty}\frac{\partial^2}{\partial t^2}\left(e^{j\omega t}\right)\cdot\left[P_+(k)e^{-j2\pi kx} + P_-(k)e^{j2\pi kx}\right]dk = \\ = c^2\int_{-\infty}^{+\infty} e^{j\omega t}\cdot\frac{\partial^2}{\partial x^2}\left[P_+(k)e^{-j2\pi kx} + P_-(k)e^{j2\pi kx}\right]dk \\ \Downarrow \\ -\int_{-\infty}^{+\infty}\omega^2 e^{j\omega t}\cdot X(x;k)\,dk \\ = \int_{-\infty}^{+\infty} c^2 e^{j\omega t}\cdot X''(x;k)\,dk \end{gathered} \tag{6}$$

In the above, the following substitution has been performed.

$$\begin{gathered} X(x;k) = \left[P_+(k)e^{-j2\pi kx} + P_-(k)e^{j2\pi kx}\right] \\ \Downarrow \\ X'' = \frac{\partial^2}{\partial x^2}X(x;k) = -4\pi^2k^2X \end{gathered} \tag{7}$$

Therefore, the only requirement for function is to satisfy the boundary conditions. Commonly boundary conditions are provided as specifications for the reflection coefficient R as function of frequency or more rarely wavenumber. For example, if totally reflective boundary conditions are considered, the following need to be considered.

$$\begin{gathered} \Gamma(\omega @ x = 0) = \frac{\left|P_+(k)e^{-j2\pi k\cdot 0}\right|}{\left|P_-(k)e^{+j2\pi k\cdot 0}\right|} = 1 \\ \Gamma(\omega @ x = L) = \frac{\left|P_-(k)e^{+j2\pi k\cdot L}\right|}{\left|P_+(k)e^{-j2\pi k\cdot L}\right|} = 1 \end{gathered} \tag{8}$$

In effect, and by taking into account conservation of energy, the following facts are obtained.

$$P_+(k) = -P_-(k), \forall k \tag{9}$$

$$e^{j4\pi kL} = 1 \Rightarrow 4\pi kL = n\cdot 2\pi \Rightarrow k = k_n = \frac{n}{2L} \tag{10}$$

$$\begin{gathered} X(x;k) = X(x;k_n) = X_n(x) = \\ = P_n\left[e^{-j2\pi k_n x} - e^{+j2\pi k_n x}\right] = 2jP_n\sin\frac{n\pi x}{L} \end{gathered} \tag{11}$$

$$p(x;t) = 2j\sum_{n=-\infty}^{+\infty} P_n e^{j\omega_n t}\sin\frac{n\pi x}{L}, \omega_n = c\cdot 2\pi k_n = \frac{n\pi c}{L} \tag{12}$$

In the general case, where neither the boundary conditions nor the shape of the domain is 1D and/or standard, modal analysis can still be applied and the eigen-shapes be obtained by numerical methods, e.g. the Complex Singular Value (cSVD) decomposition [1]. Therefore, in general an expression as follows can be assumed for the sound pressure within the domain of interest.

$$p(x;t)=\sum_{n=-\infty}^{+\infty}P_n e^{j\omega_n t}\Phi_n(x)$$
$$\omega_n=c\cdot 2\pi k_n=\frac{n\pi c}{L},0<x<L \qquad (13)$$

Returning now to the non-homogeneous case of interest and following standard procedure, the following expansions are obtained.

$$p(x;x_S;t)=e^{j\omega_S t}\sum_{n=-\infty}^{+\infty}P_n(x_S(t))\Phi_n(x),0<x<L\Rightarrow$$
$$\Rightarrow\begin{cases}\dfrac{\partial^2 p}{\partial t^2}=e^{j\omega_S t}\displaystyle\sum_{n=-\infty}^{+\infty}\begin{bmatrix}\dfrac{\partial P_n}{\partial x_S}\ddot{x}_S+\dfrac{\partial^2 P_n}{\partial x_S^2}\dot{x}_S^2+\\+2j\omega_s\dfrac{\partial P_n}{\partial x_S}\dot{x}_S-\\-\omega_S^2 P_n(x_S)\end{bmatrix}\Phi_n(x)\\ \dfrac{\partial^2 p}{\partial x^2}=e^{j\omega_S t}\displaystyle\sum_{n=-\infty}^{+\infty}P_n(x_S(t))\dfrac{\partial^2\Phi_n}{\partial x^2}\end{cases} \qquad (14)$$

For simplifying the problem at hand, the velocity and acceleration of the source will be considered relatively small.

$$\frac{\partial^2 p}{\partial t^2}\approx-\omega_S^2 e^{j\omega_S t}\sum_{n=-\infty}^{+\infty}P_n(x_S(t))\Phi_n(x) \qquad (15)$$

The homogeneous boundary conditions need to be employed at this point.

$$\frac{\partial^2\Phi_n}{\partial x^2}=-4\pi^2k_n^2\Phi_n(x) \qquad (16)$$

Finally, it can be seen that if the eigenshapes of the homogeneous problem consist a base of the Hilbert space. Therefore, the following decomposition for the excitation can be obtained.

$$m(t)\cdot\delta(x-x_S(t))=$$
$$=e^{j\omega_S t}\sum_{n=-\infty}^{+\infty}\Phi_n(x_S(t))\Phi_n(x),0<x<L \qquad (17)$$

By merging the above expression, in the original wave equation with a wandering source, the following is obtained.

$$\left.\begin{aligned}e^{j\omega_S t}\sum_{n=-\infty}^{+\infty}\left[4\pi^2k_n^2-\omega_S^2\right]P_n(x_S(t))\Phi_n(x)=\\=e^{j\omega_S t}\sum_{n=-\infty}^{+\infty}\Phi_n(x_S(t))\Phi_n(x)\end{aligned}\right\}\Rightarrow$$
$$\Rightarrow P_n(x_S(t))=\frac{\Phi_n(x_S(t))}{\omega_n^2-\omega_S^2} \qquad (18)$$

Therefore, the sound pressure received by the receiver obtains the form.

$$p(x_R;x_S(t);t)=e^{j\omega_S t}\sum_{n=-\infty}^{+\infty}\frac{\Phi_n(x_R)}{\omega_n^2-\omega_S^2}\Phi_n(x_S(t))$$
$$\omega_n=\frac{n\pi c}{L},0<x<L \qquad (19)$$

As seen in the above, the sound source position signal is propagated through a nonlinear operator in order to determine the sound pressure level signal at the receiver's location. In the general case, of non-standard domains of complicated geometry, complicated boundary (and possibly initial) conditions, noise and interference etc, the nonlinear operator connecting TX position and RX sound pressure will include also time derivatives of signal x_S(t). For example, even in the case considered earlier if velocity and acceleration of the source are not negligible then the expression for sound pressure will still contain the time derivatives of source displacement.

Sound source motion estimation

In this context, simulation can be used to determine the Volterra kernels, or equivalently Wiener G-functionals, of the operator connecting source displacement xS(t) to sound pressure at the receiver p(t). As explained in the sequel, identification can be performed by applying a technique combining cSVD and the Volterra theory to either numerical simulation results and/or field recorded data series [1,3].

In effect, this can be done by using the Feedback System Inversion Lemma. In Figure 2 the operator interconnecting the source displacement signal to the receiver acoustic pressure one as well as its inverse are shown.

In Figure 2, the left-hand side is a block-diagram representation of the operator interconnecting the source displacement with acoustic pressure. The operator includes a linear and a nonlinear part as argued earlier. The nonlinear part may be written in the following form of a Volterra series [2].

$$\hat{p}(t@x_R)=\mathbf{G}\left[x_S(t)\right]=\sum_{m=0}^{+\infty}p_m(t),$$
$$p_m(t)=\iint\cdots\int\begin{Bmatrix}g_m(t_1,t_2,\ldots,t_m)\cdot\\ \cdot x_S(t-t_1)x_S(t-t_2)\cdots x_S(t-t_m)\cdot\\ \cdot dt_1dt_2\cdots dt_m\end{Bmatrix} \qquad (20)$$

In the above, stands for the m-th (nonlinearity) degree Volterra kernel. Note the similarity of the Volterra kernels to the impulse response of linear systems. In effect, the receiver can use this information to process the acoustic pressure signal by means of the feedback structure of the right-hand side in Figure 2. This will allow generating an estimate of the sound source position.

The real-time motion estimation processing scheme can be formulated as the following state-space (phase-plane) description. The description's state vector is by assumption partitioned as follows [3].

$$\dot{\mathbf{x}}_1=\mathbf{A}\cdot\mathbf{x}_1+\boldsymbol{\psi}(\mathbf{y}_2)+\mathbf{d} \qquad (21)$$

$$\dot{\mathbf{x}}_2=\mathbf{F}(\mathbf{x}_1)\cdot\mathbf{x}_2+\mathbf{G}(\mathbf{x}_1)\cdot\mathbf{u}$$
$$\mathbf{y}_2=\mathbf{C}_2\cdot\mathbf{x}_2 \qquad (22)$$

In the above

$$\mathbf{x}_1, \mathbf{d} \in \mathbb{R}^{n_1}, \mathbf{A} \in \mathbb{R}^{n_1 \times n_1}$$
$$\mathbf{x}_2 \in \mathbb{R}^{n_2}, \mathbf{u} \in \mathbb{R}^{m_2}, \mathbf{y}_2 \in \mathbb{R}^{p_2}, \mathbf{C}_2 \in \mathbb{R}^{p_2 \times n_2}$$
$$\boldsymbol{\psi} : \mathbf{y}_2 \in \mathbb{R}^{p_2} \to \boldsymbol{\psi}(\mathbf{y}_2) \in \mathbb{R}^{n_2}$$
$$\mathbf{F} : \mathbf{x}_1 \in \mathbb{R}^{n_1} \to \mathbf{F}(\mathbf{x}_1) \in \mathbb{R}^{n_2 \times n_2}$$
$$\mathbf{G} : \mathbf{x}_1 \in \mathbb{R}^{n_1} \to \mathbf{G}(\mathbf{x}_1) \in \mathbb{R}^{n_2 \times m_2}$$

The partitioning of the n-dimensional state vector $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T \end{bmatrix}^T$ to two components $\mathbf{x}_1$ (dimension n_1) and $\mathbf{x}_2$ (dimension n_2) whose dynamics are determined by the equations (21) and (22) respectively is reflecting the partitioning of the system to a low-pass (LP) and a band-pass (BP) part.

Evidently, the LP part corresponds to the wandering source's motion. Typically it includes the source's position and velocity. The BP part depicts the sound wave propagation through the medium and contain a truncated set of modal dynamical variables as those identified in equation (13). The estimator of the source's motion variables at the receiver is employing the received acoustic signal to generate its estimates. This is manifested by the nonlinear term $\psi(\mathbf{y}_2)$ in equation (21). This term has to be a nonlinear one since the estimator needs to bring the BP received signal down to baseband and utilize the modulating envelope for generating its estimates (see equation (19)). The vector $\mathbf{y}_2$ includes the observables of the acoustic signal transmission; commonly it is a scalar corresponding to the received sound pressure. Finally, consistency check of the motion variable estimates, x_1, is performed by using them to generate acoustics related signals in equations (22).

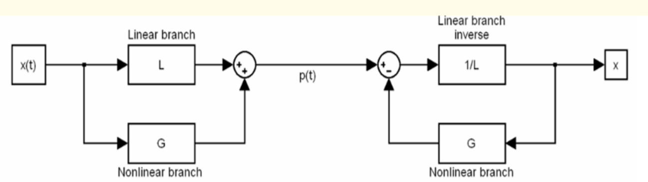

Figure 2: Feedback System Inversion Lemma formulation.

This rationale is made clearer if the definitions of LP and BP systems are provided. For Single-Input-Single-Output (SISO) linear, asymptotically stable systems, whose dynamics can be defined by the use of scalar transfer functions obtaining no poles in the right-half s-plane or on the imaginary axis, the following LP system definition is given.

$$\forall \varepsilon > 0, \exists W(\varepsilon) > 0 : |\omega| > W(\varepsilon) \Rightarrow |H(\omega)| < \varepsilon \qquad (23)$$

In the above $H(\omega) = H(s = j\omega = j2\pi f)$ stands for the transfer function of the SISO system. Parameter ($2W$) when $\varepsilon = \frac{\|H(\omega)\|_\infty^2}{2} = \frac{\left[\sup_{\omega \in \mathbb{R}}(H(\omega))\right]^2}{2}$ is called half-power band width or 3dB bandwidth (or simply bandwidth if there is no possibility of confusion) of the system; it is denoted as *BW*

Generalization of the LP system gives rise to the BP system. For Single-Input-Single-Output (SISO) linear, asymptotically stable systems, the following BP system definition is given.

$$\exists \omega_c, \forall \varepsilon > 0, \exists W(\varepsilon) > 0 : |\omega \pm \omega_c| > W(\varepsilon) \Rightarrow |H(\omega)| < \varepsilon \qquad (24)$$

In analogy, bandwidth *BW* is defined for BP systems. Circular frequency ω_c is called carrier frequency of the system. Evidently, LP systems are BP systems with ω_c=0 .

In the case of linear, asymptotically stable, Multi-Input-Multi-Output (MIMO) systems the above definitions may be easily generalized, with respect to the transfer function matrix $\mathbf{H}(s)$ of the system and its frequency-dependent maximum singular value $\sigma_{\max}\{\mathbf{H}(\omega)\}$ For example, the BP system definition may be extended in the MIMO case as follows.

$$\begin{aligned} &\exists \omega_c, \forall \varepsilon > 0, \exists W(\varepsilon) > 0 : \\ &\quad |\omega \pm \omega_c| > W(\varepsilon) \Rightarrow \sigma_{\max}\{\mathbf{H}(\omega)\} < \varepsilon \end{aligned} \qquad (25)$$

Finally, it is mentioned the definitions of BP and LP can be easily extended to finite energy, scalar or vector signals.

The following Taylor expansions around zero for the multivariable vector function ψ and the multivariable matrix functions **F,G** will be presumed.

$$\begin{matrix}\boldsymbol{\psi} \\ \mathbf{F} \\ \mathbf{G}\end{matrix}(\xi) = \begin{matrix}\boldsymbol{\psi}_0 \\ \mathbf{F}_0 \\ \mathbf{G}_0\end{matrix} + \begin{matrix}\boldsymbol{\psi}_1 \\ \mathbf{F}_1 \\ \mathbf{G}_1\end{matrix}(\xi) + \begin{matrix}\boldsymbol{\psi}_2 \\ \mathbf{F}_2 \\ \mathbf{G}_2\end{matrix}(\xi \otimes \xi) + \begin{matrix}\boldsymbol{\psi}_3 \\ \mathbf{F}_3 \\ \mathbf{G}_3\end{matrix}(\xi \otimes \xi \otimes \xi) + \cdots \quad (26)$$

In the above, $\otimes$ denotes Kronecker vector product and the sho rthand notation $\xi^{\otimes k}$ will be used to denote the k-th power of vector ξ in the Kronecker product sense.

The procedure of producing the terms in the expansion above is carried out in an element-wise manner. In specific, it is demonstrated in the case of matrix function F but can be straightforwardly generalized for ψ and G.

$$\begin{aligned} &\mathbf{F}(\mathbf{x}_1) = \left[f_{ij}(\mathbf{x}_1)\right], 1 \le i, j \le n_2 \\ &\qquad\qquad \Downarrow \\ &\mathbf{F}_k\left(\mathbf{x}_1^{\otimes k}\right) = \left[\mathbf{f}_{ij}^{(k)} \cdot \mathbf{x}_1^{\otimes k}\right], 1 \le i, j \le n_2 \end{aligned} \qquad (27)$$

In the above, the k-th "derivative" vector, $\mathbf{f}_{ij}^{(k)}$ is set so that

$$\mathbf{x}_1^{\otimes k} \cdot \mathbf{f}_{ij}^{(k)} = \left\{\frac{\mathbf{x}_1}{k!} \cdot \frac{\partial}{\partial \mathbf{x}_1}\right\}^k f_{ij}(\mathbf{x}_1 = \mathbf{0})\Big| \qquad (28)$$

In order to match the expansion of the operator with respect to the Kronecker vector power $\mathbf{x}_1^{\otimes k}$; evidently, the partial derivatives , appearing in the operator expansion, are evaluated at $\mathbf{x}_1$=0 and some of the elements of $\mathbf{f}_{ij}^{(k)}$ nay need to be set to zero. Special cases are k = 0 and k = 1.

$$\begin{aligned} &\mathbf{F}_0 = \mathbf{F}(\mathbf{0}) = \left[f_{ij}(\mathbf{x}_1 = \mathbf{0})\right], 1 \le i, j \le n_2 \\ &\mathbf{F}_1(\mathbf{x}_1) = \left[\mathbf{x}_1 \cdot \nabla f_{ij}(\mathbf{x}_1 = \mathbf{0})\right], 1 \le i, j \le n_2 \end{aligned} \qquad (29)$$

Note that, as in the general case, the elements of **F1** are inner products of the Kronecker power vector for k = 1 times the element-wise partial derivative (grad) vector evaluated at $\mathbf{x}_1$=0.

By using the expansion in (26) for matrix functions **F,G,** equations (22) are transformed to the following.

$$\dot{\mathbf{x}}_2 = \mathbf{F}_0 \cdot \mathbf{x}_2 + \underbrace{\left[\mathbf{G}_0 \quad \mathbf{\Gamma}_0\right]}_{\mathbf{\Gamma}} \cdot \begin{bmatrix} \mathbf{u} \\ \mathbf{r} \end{bmatrix} \qquad (30)$$
$$\mathbf{y}_2 = \mathbf{C}_2 \cdot \mathbf{x}_2$$

In the above, auxiliary signal vector **r** is defined as follows.

$$\mathbf{r}(t) \triangleq \sum_{k=1}^{\infty} \left\{ \mathbf{F}_k \left(\mathbf{x}_1^{\otimes k}\right) \cdot \mathbf{x}_2 + \mathbf{G}_k \left(\mathbf{x}_1^{\otimes k}\right) \cdot \mathbf{u} \right\} \qquad (31)$$

Furthermore, it holds that
$\mathbf{\Gamma} \in \mathbb{R}^{n_2 \times (m_2 + n_2)}, \mathbf{\Gamma}_0 \in \mathbb{R}^{n_2 \times n_2}$

Γ_0 is a square binary matrix (i.e. (0,1)-matrix) accounting for the fact that certain elements of signal vector r may be identically zero.

With respect to the formulation of equations (30), in the sequel systems of the form (22) will be considered with the additional assumption that the following transfer function matrix is BP around a carrier frequency ω_c.

$$\mathbf{H}_2(s) = \mathbf{C}_2 \cdot (s\mathbf{I} - \mathbf{F}_0)^{-1} \cdot \mathbf{\Gamma} \qquad (32)$$

The coupling of such systems to systems of the form (21) will be investigated with the additional assumption that the following transfer function matrix is LP.

$$\mathbf{H}_1(s) = (s\mathbf{I} - \mathbf{A})^{-1} \qquad (33)$$

Before the analysis, a brief presentation of the Hilbert Transform and associated mathematical apparatus is needed.

The hilbert transform and the complex envelope of a bp signal

The Hilbert Transform allows analyzing phase selective and spectrally decoupled systems which interact through amplitude or phase modulation. The Hilbert Transform, in contrast to the Laplace and Fourier Transforms, does not establish a new domain but both its input and output are in the time domain. In literature [4-7], the Hilbert Transform is often seen as a $\pm\pi/2$ phase shift applied to a scalar or vector signal. A more formal definition as an integral transform is given below for a signal vector.

$$\hat{\mathbf{x}}(t) \triangleq \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{t - t_1} \mathbf{x}(t_1) dt_1 = \frac{1}{\pi t} * \mathbf{x}(t)$$
$$\Updownarrow \qquad (34)$$
$$\hat{\mathbf{x}}(f) = -j\,\mathrm{sgn}(f)\mathbf{x}(f)$$

In the above, the star (*) stands for the convolution operator. Specifically, the definition of the convolution operator, for two real or complex functions of one, real variable, is the following.

$$\begin{aligned}\phi(\chi) = \phi_1(\chi) * \phi_2(\chi) &\triangleq \\ &\triangleq \int_{-\infty}^{+\infty} \phi_1(\xi)\phi_2(\chi - \xi) d\xi = \\ &= \int_{-\infty}^{+\infty} \phi_2(\xi)\phi_1(\chi - \xi) d\xi = \phi_2(\chi) * \phi_1(\chi)\end{aligned} \qquad (35)$$

The sign function is defined below.

$$\mathrm{sgn}(\xi) = \begin{cases} +1, \xi > 0 \\ 0, \xi = 0 \\ -1, \xi < 0 \end{cases} \qquad (36)$$

The Fourier transform to be used in this text is the bilateral Fourier integral given below for a signal vector.

$$\mathbf{x}(f) \triangleq \int_{-\infty}^{+\infty} \mathbf{x}(t) \exp(-j2\pi ft) dt$$
$$\Updownarrow \qquad (37)$$
$$\mathbf{x}(t) = \int_{-\infty}^{+\infty} \mathbf{x}(f) \exp(j2\pi ft) df$$

The Hilbert transform is linear and possesses an inverse transform, defined as follows.

$$\mathbf{x}(t) \triangleq -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{1}{t - t_1} \hat{\mathbf{x}}(t_1) dt_1 \qquad (38)$$

Two important properties make the Hilbert transform extremely useful in the analysis of BP signals and systems especially when amplitude modulation is involved. The first one concerns the product of a BP and a LP signal; the scalar case is given below.

$$\left.\begin{aligned} \chi(t) &= \chi_1(t)\chi_2(t) \\ \chi_1(t) &\in \mathbb{R} \text{ is LP} \\ \chi_2(t) &\in \mathbb{R} \text{ is BP} \end{aligned}\right\} \Rightarrow \hat{\chi}(t) = \chi_1(t)\hat{\chi}_2(t) \qquad (39)$$

The sinusoidal signals are linked as phase shifts by the Hilbert transform.

$$\begin{aligned} \chi(t) = \cos(2\pi f_0 t) &\Rightarrow \hat{\chi}(t) = \sin(2\pi f_0 t) \\ \chi(t) = \sin(2\pi f_0 t) &\Rightarrow \hat{\chi}(t) = -\cos(2\pi f_0 t) \end{aligned} \qquad (40)$$

Using the Hilbert transform, the pre-envelope, $\mathbf{x}_+(t)$, of a signal vector, **x**(t), is defined as a signal vector with elements in $\mathbb{C}$ as follows.

$$\mathbf{x}_+(t) \triangleq \mathbf{x}(t) + j\hat{\mathbf{x}}(t)$$
$$\Updownarrow \qquad (41)$$
$$\mathbf{x}_+(f) = \mathbf{x}(f) + j(-j\,\mathrm{sgn}(f)\mathbf{x}(f)) = \begin{cases} 2\mathbf{x}(f), f > 0 \\ \mathbf{x}(f), f = 0 \\ 0, f < 0 \end{cases}$$

As can be seen the pre-envelope is a complex signal vector allowed to have nonzero spectrum only in non-negative frequencies. Using the pre-envelope, $\mathbf{x}_+(t)$, the complex envelope, $\tilde{\mathbf{x}}(t)$, is defined for a BP signal vector, **x**(t), with carrier frequency $\omega_c = 2\pi f_c$.

$$\mathbf{x}_+(t)=\tilde{\mathbf{x}}(t)\exp(j2\pi f_c t)$$
$$\Updownarrow$$
$$\tilde{\mathbf{x}}(t)\triangleq\mathbf{x}_+(t)\exp(-j2\pi f_c t) \qquad (42)$$
$$\Updownarrow$$
$$\tilde{\mathbf{x}}(f)=\mathbf{x}_+(f-f_c)=2\mathbf{x}(f-f_c)$$

The last one of the above relationships in combination with the fact that signal **x**(t) is BP with carrier frequency $\omega_c=2\pi f_c$ leads to the conclusion that the complex envelope of a BP signal is an LP signal.

The complex envelope may be decomposed to a real and an imaginary component.

$$\tilde{\mathbf{x}}(t)=\mathbf{x}_C(t)+j\mathbf{x}_S(t) \qquad (43)$$

In telecommunications literature [4,5] the real component, $\mathbf{x}_c$(t) , is referred to as the in-phase (or I for short) component and the imaginary component, $\mathbf{x}_s$(t) , is referred to as the quadrature (or Q for short) component. Clearly, the I and Q components of the complex envelope are mutually orthogonal and preserve the complete information content of the BP signal from which they are generated. Furthermore, as can be seen from the first one of equations (42), the complex envelope is a generalization of the concept of amplitude modulation applied to the generalized imaginary exponential carrier signal $\exp(j\omega_c t)$. The following algebraic manipulation supports this claim.

$$\mathbf{x}(t)=\mathrm{Re}\left[\mathbf{x}_+(t)\right]=\mathrm{Re}\left[\tilde{\mathbf{x}}(t)\exp(j2\pi f_c t)\right]$$
$$\Downarrow \qquad (44)$$
$$\mathbf{x}(t)=\mathbf{x}_C(t)\cos(\omega_c t)-\mathbf{x}_S(t)\sin(\omega_c t)$$

The above clearly demonstrates that the original (real) signal vector is generated as amplitude modulation of carrier signal $\cos(\omega_c t)$ by the I component and amplitude modulation of carrier signal $\sin(\omega_c t)$ by the Q component. Note that the two carriers, although demonstrating the same frequency, are mutually orthogonal.

The equivalent LP system of BP-LP nonlinear coupling

We will now return to the coupled system whose dynamics are given in equations (21) and (22) by taking into account that the transfer function matrix in equation (32) is BP and the one in (33) is LP. Then, an equivalent exclusively LP system may be obtained, by using the complex envelope of the BP signal vector $\mathbf{x}_2$(t).

At first, due to assumption (33) state vector $\mathbf{x}_1$(t) is LP, as it is generated by an LP system. Indeed, assuming zero initial conditions, the state vector will be given by the following in the frequency domain.

$$\mathbf{x}_1(\omega)=\mathbf{H}_1(\omega)\cdot\mathbf{d}_1(\omega) \qquad (45)$$

In the above, the auxiliary signal $\mathbf{d}_1$(t) is defined as follows.

$$\mathbf{d}_1(t)=\boldsymbol{\psi}(\mathbf{y}_2(t))+\mathbf{d}(t) \qquad (46)$$

Therefore, if $\mathbf{d}_1$(t) is assumed to be a random signal of the white Gaussian noise type, i.e. possessing power spectral density constant over all frequencies, $\mathbf{x}_1$(t) will comply to the requirement of equation (23).

The next step is to observe that an arbitrary Kronecker power of an LP signal vector is also LP with possibly larger *BW*. Therefore, the elements of matrices **F,G,** if viewed as scalar signals are LP, at least in the case that the expansions in equation (26) obtain a finite number of terms. Property (40) guarantees, then, that the following holds.

$$\dot{\mathbf{x}}_2=\mathbf{F}(\mathbf{x}_1)\cdot\mathbf{x}_2+\mathbf{G}(\mathbf{x}_1)\cdot\mathbf{u}$$
$$\Downarrow \qquad (47)$$
$$\dot{\hat{\mathbf{x}}}_2=\mathbf{F}(\mathbf{x}_1)\cdot\hat{\mathbf{x}}_2+\mathbf{G}(\mathbf{x}_1)\cdot\hat{\mathbf{u}}$$

In the above the following fact for the time derivative of a signal vector has been used.

$$\hat{\dot{\mathbf{x}}}(t)=\dot{\hat{\mathbf{x}}}(t) \qquad (48)$$

This is a direct consequence of the linearity property of the Hilbert transform. If the second equation in (47) is multiplied by *j* and then added to the first, the following dynamic equation for the pre-envelope, $\mathbf{x}_{2+}$, of the state vector signal $\mathbf{x}_2$ is obtained.

$$\dot{\mathbf{x}}_{2+}=\mathbf{F}(\mathbf{x}_1)\cdot\mathbf{x}_{2+}+\mathbf{G}(\mathbf{x}_1)\cdot\mathbf{u}_+ \qquad (49)$$

Because of the assumption that the transfer function in equation (32) is BP with carrier frequency $\omega_c=2\pi f_c$ the pre-envelope $\mathbf{x}_{2+}$, of the state vector signal $\mathbf{x}_2$ may be expressed as a product between a modulating complex envelope factor, $\tilde{\mathbf{x}}_2$, and an imaginary exponential signal acting as a generalized sinusoidal carrier signal.

$$\mathbf{x}_{2+}(t)=\tilde{\mathbf{x}}_2(t)\exp(j\omega_c t)$$
$$\Downarrow \qquad (50)$$
$$\dot{\mathbf{x}}_{2+}(t)=\dot{\tilde{\mathbf{x}}}_2(t)\exp(j\omega_c t)+j\omega_c\tilde{\mathbf{x}}_2(t)\exp(j\omega_c t)$$

If the input signal vector u is assumed to be tuned to the BP system, then it may be expressed similarly.

$$\mathbf{u}_+(t)=\tilde{\mathbf{u}}(t)\exp(j\omega_c t) \qquad (51)$$

By substituting equations (50) and (51) in equation (49) the following relationship is finally obtained for the complex envelope of the BP system's state vector.

$$\dot{\tilde{\mathbf{x}}}_2=\left[\mathbf{F}(\mathbf{x}_1)-j\omega_c\mathbf{I}\right]\cdot\tilde{\mathbf{x}}_2+\mathbf{G}(\mathbf{x}_1)\cdot\tilde{\mathbf{u}} \qquad (52)$$

If a similar expression is adopted for the output vector, $\mathbf{y}_2$ is adopted then one obtains the following expression for its complex envelope $\tilde{\mathbf{y}}_2$ by substituting the first equation (50) in the second equation (22).

$$\left.\begin{matrix}\mathbf{y}_{2+}(t)=\tilde{\mathbf{y}}_2(t)\exp(j\omega_c t)\\ \mathbf{y}_2(t)=\mathrm{Re}\left\{\mathbf{y}_{2+}(t)\right\}\end{matrix}\right\}\Rightarrow\tilde{\mathbf{y}}_2=\mathbf{C}_2\cdot\tilde{\mathbf{x}}_2 \qquad (53)$$

By using the above, one can rewrite equation (21) so that it contains the I and Q components of $\tilde{\mathbf{y}}_2$, $\mathbf{y}_{2c}$ and $\mathbf{y}_{2s}$ respectively. This is done by employing directly expression (44) in equation (21). However, further simplification is possible by exploiting the assumption that transfer function (33) is LP in order to eliminate high-frequency terms, i.e. terms including a "carrier" factor of the form $\exp(\pm jk\omega_c t), k = 1,2,3,\ldots$. Such factors appear as a direct consequence of the fact that the coupling term in equation (21) is the nonlinear function ψ. Then, terms with factors of the form $\exp(\pm jk\omega_c t), k = 1,2,3,\ldots$ may be neglected on the basis of the spectral decoupling between the LP and the BP system. This is translated to the requirement that the LP system's BW is much smaller than the BP system's carrier frequency $\omega_c = 2\pi f_c$.

By rewriting equations (26) and (27) in the case of multivariable vector function $\psi(\mathbf{y}_2)$ it is obtained that

$$\boldsymbol{\psi}(\mathbf{y}_2) = \left[\sum_{k=0}^{\infty} \boldsymbol{\psi}_i^{(k)} \cdot \mathbf{y}_2^{\otimes k}\right], \boldsymbol{\psi}_i^{(0)} = \psi_i(\mathbf{y}_2 = \mathbf{0}), 1 \le i \le n_1 \quad (54)$$

An LP contribution from the above is possible only if k is even. Indeed, an LP contribution is this part of each term in the above expansion which is not multiplied by a carrier factor $\exp(\pm jk\omega_c t)$, $k = 1,2,3,\ldots$. Therefore, despite the 0-th term, which obviously participates in the LP part of the signal vector in equation (54), all terms for strictly positive k contain a component multiplied by $\exp(\pm jk\omega_c t), k = 1,2,3,\ldots$. However, when k is odd only this component is present; e.g. for $k = 1$. On the other hand, when k is even, except the carrier-multiplied component, there exists a carrier-free one, too. This is made clearer with an example, e.g. for $k = 2$.

$$\begin{aligned}\mathbf{y}_2^{\otimes 2} &= \left(\mathrm{Re}\left[\tilde{\mathbf{y}}_2 e^{j\omega_c t}\right]\right)^{\otimes 2} = \left(\frac{\tilde{\mathbf{y}}_2 e^{j\omega_c t} + \tilde{\mathbf{y}}_2^* e^{-j\omega_c t}}{2}\right)^{\otimes 2} = \\ &= \frac{\tilde{\mathbf{y}}_2 \otimes \tilde{\mathbf{y}}_2^*}{2} + \left(\frac{\tilde{\mathbf{y}}_2}{2}\right)^{\otimes 2} e^{j2\omega_c t} + \left(\frac{\tilde{\mathbf{y}}_2^*}{2}\right)^{\otimes 2} e^{-j2\omega_c t} = \\ &= \frac{\mathbf{y}_{2C}^{\otimes 2} + \mathbf{y}_{2S}^{\otimes 2}}{2} + \left(\frac{\tilde{\mathbf{y}}_2}{2}\right)^{\otimes 2} e^{j2\omega_c t} + \left(\frac{\tilde{\mathbf{y}}_2^*}{2}\right)^{\otimes 2} e^{-j2\omega_c t}\end{aligned} \quad (55)$$

In the above, only the first term is an LP one as the other two contain a carrier factor.

In conclusion, the equations (21) and (22) of coupled subsystems, which obtain LP and BP transfer function matrices as in equations (33) and (32), respectively, can be reduced to the following LP equivalent system of equations:

$$\dot{\mathbf{x}}_1 = \mathbf{A}\cdot\mathbf{x}_1 + \boldsymbol{\psi}_{LP}(\tilde{\mathbf{y}}_2) + \mathbf{d} \quad (56)$$

$$\begin{aligned}\dot{\tilde{\mathbf{x}}}_2 &= \left[\mathbf{F}(\mathbf{x}_1) - j\omega_c\mathbf{I}\right]\cdot\tilde{\mathbf{x}}_2 + \mathbf{G}(\mathbf{x}_1)\cdot\tilde{\mathbf{u}} \\ \tilde{\mathbf{y}}_2 &= \mathbf{C}_2\cdot\tilde{\mathbf{x}}_2\end{aligned} \quad (57)$$

In the above, all signals in the BP subsystem have been substituted by their LP complex envelopes, e.g. $\mathbf{x}_2(t)$ by $\tilde{\mathbf{x}}_2(t) = \mathbf{x}_{2C}(t) + j\mathbf{x}_{2S}(t)$ Furthermore, in the LP subsystem equation, multivariable vector function has been substituted by the LP one $\boldsymbol{\psi}_{LP}(\tilde{\mathbf{y}}_2) = \boldsymbol{\psi}_{LP}(\mathbf{y}_{2C}, \mathbf{y}_{2S})$ which is produced by omitting from the original all odd terms and the modulated carrier parts of the even terms. The main benefit in using the description of equations (56) and (57) instead of the original ones in equations (21) and (22) is that, because it is LP but otherwise grasps all the essential dynamics of the subsystem at hand, the time step for the integration of the dynamical equations may be much smaller than in the original one. For example, in typical underwater acoustic applications, as the one examined in this text, the *BW* of both the LP and the BP system is commonly in the order of magnitude of 1 kHz. However, the carrier frequency may be 10 kHz or even larger. Therefore, the integration step may be increased at least two order of magnitudes; such a possibility makes investigations much easier. Another benefit is that by using the equivalent LP system the carrier frequency, which does not play such a significant role in the understanding of the dynamics, comes into the analysis as a simple parameter. In effect, as far as the main assumptions are satisfied the selection of the carrier frequency does not affect any significant conclusions for the behavior of the system at hand.

Finally, as will be presented in future work, many methodologies that can be carried out at the original system may be applied to the LP equivalent, but with the benefit of faster integration. The actual BP signals, after the solution is determined for their LP equivalents, are generated by modulating the carrier with the complex envelope.

Conclusion

A signal processing scheme allowing an underwater acoustic silent receiver to estimate the motion variables of a wandering acoustic source, emitting at a single frequency is presented. The processing is formulated as a coupled bandpass-lowpass nonlinear system by employing the Hilbert transform. The bandpass part depicts underwater sound propagation, while the lowpass one the motion of the source. Such structure and behavior are commonly observed in underwater acoustic source localization problem in confined domains like e.g. the coastal zone. The result of the proposed methodology is a lowpass equivalent system which depicts the essential dynamics of the interaction between the modulating envelope of the bandpass subsystem signal with the lowpass subsystem signal. This description may prove of value in military or civilian applications, especially if combined with approaches to tackle uncertainty in either the domain's boundary conditions or the transmitted acoustic signal characteristics.

The problem is first reformulated in an amplitude-modulated framework which is valid for moderate source speed and acceleration values. Then, the nonlinear signal processing system is presented. In a future work, a digital system will be presented to implement the concept. Results from data series currently being collected will also be presented for benchmarking.

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