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Review Article

Hamiltonians that Generate Coherent States as their Ground States

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Abstract

We uncover which is the Hamiltonian that has a coherent state as its ground state. Thus, we explain here how coherent states (CS) become eigenfunctions of a displaced quadratic potential V (x) that depends on the CS amplitude α. For each amplitude we have a different V (x). The associated ground state is a coherent state.

Keywords: Coherent States; Exact Analytic Form; Potential Function

Introduction

We begin our review by briefly recapitulating ideas revolving around coherent states of the harmonic oscillator (HO) |α >, or Glauber states [1-3]. A coherent state (CS) |α >is a specific kind of quantum state of minimum uncertainty. Indeed, the one that most resembles a classical state. In general, α is a complex variable.

CS are applicable to the quantum harmonic oscillator, the electromagnetic field, etc., and portray a maximal type of coherence and a classical sort of behavior. The states α are normalized, i.e., α α = 1, and they yield a resolution of the identity operator

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = 1, \quad \text{-------------------(1)}$$

Which is a completeness relation for the coherent states [3]. The standard coherent states α for the harmonic oscillator are eigenstates of the annihilation operator aˆ, with complex eigenvalues

$$\alpha = \frac{q+ip}{\sqrt{2}}, \quad \text{-------------(2)}$$

Which satisfy aˆ|α⟩ = α|α⟩ [3].

The n−th HO eigenfunction is

$$\mathcal{H}_n(x) = \left(\pi^{\frac{1}{2}} 2^n n!\right)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_n(x), \quad \text{--------------(3)}$$

Where H_n is Hermite's n−th order generalized function

$$\phi_n(x) = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{4}} \mathcal{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \quad \text{-------------(4)}$$

While H_n is the concomitant Hermite polynomial. In the x-representation, the coherent state reads

$$\psi_\alpha(x) = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \phi_n(x), \quad \text{--------------(5)}$$

Or

$$\psi_\alpha(x) = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{4}} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mathcal{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right). \quad \text{------------(6)}$$

For notational convenience we set below $\sqrt{\frac{m\omega}{\hbar}} = 1$. Thus, for the HO we have

$$\phi_n(x) = \mathcal{H}_n(x), \quad \text{-----------(7)}$$

And for its coherent states (CS)

$$\psi_\alpha(x) = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \mathcal{H}_n(x)\,. \qquad \text{----------(8)}$$

Closed analytic form for CS in the x representation

Now we address an interesting fact: the CS can be given a compact analytic form in the x representation of quantum mechanics, as discussed in [5].

It is well known the annihilation operator for the one-dimensional harmonic oscillator is given by

$$\hat{a} = \frac{\hat{x} + i\hat{p}}{\sqrt{2}} \qquad \text{----------(9)}$$

In the x-representation this operator is expressed as

$$\hat{a}(x) = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right) \qquad \text{--------------(10)}$$

Glauber defined coherent states as the eigenfunctions of a^(x)

$$\hat{a}(x)\psi_\alpha(x) = \frac{1}{\sqrt{2}}\left(x\psi_\alpha(x) + \frac{d\psi_\alpha(x)}{dx}\right) = \alpha\psi_\alpha(x). \qquad \text{--------(11)}$$

In another words one has

$$\frac{d\psi_\alpha(x)}{dx} = (\sqrt{2}\alpha - x)\psi_\alpha(x), \qquad \text{----------(12)}$$

A simple enough equation, whose solution becomes

$$\psi_\alpha(x) = Ce^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x}. \qquad \text{-----------(13)}$$

The constant C is fixed using the normalization requirement

$$\int_{-\infty}^{\infty} |\psi_\alpha(x)|^2 dx = |C|^2 \int_{-\infty}^{\infty} e^{-x^2} e^{\sqrt{2}(\alpha+\alpha^*)x} dx = 1. \qquad \text{-----------(14)}$$

The above equation entails that

$$\int_{-\infty}^{\infty} |\psi_\alpha(x)|^2 dx = |C|^2 e^{\frac{(\alpha+\alpha^*)^2}{2}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{\alpha+\alpha^*}{\sqrt{2}}\right)^2} dx = 1. \qquad \text{----------(15)}$$

Appeal now to the Table [6] and obtain

$$\int_{-\infty}^{\infty} e^{-\left(x - \frac{\alpha+\alpha^*}{\sqrt{2}}\right)^2} dx = \sqrt{\pi}. \qquad \text{----------(16)}$$

Accordingly,

$$C = \pi^{-\frac{1}{4}} e^{-\frac{(\alpha+\alpha^*)^2}{4}}. \qquad \text{------------(17)}$$

Note that $\alpha = \alpha_R + i\alpha_I$. Also, $\alpha + \alpha^* = 2\alpha_R$, and $\exp \alpha x = \exp \alpha_R x \exp i\alpha_I x$.

Finally, we get for $\psi_\alpha(x)$ the compact expression

$$\psi_\alpha(x) = \pi^{-\frac{1}{4}} e^{-\frac{(2\alpha_R)^2}{4}} e^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha x}, \qquad \text{----------(18)}$$

Or

$$\psi_\alpha(x) = \pi^{-\frac{1}{4}} e^{-(\alpha_R)^2} e^{-\frac{x^2}{2}} e^{\sqrt{2}\alpha_R x}, \qquad \text{----------(19)}$$

Where we omitted a phase factor $\exp i\sqrt{2}\alpha_I x$.

Properties of this special wave function are carefully reviewed in [7]. In particular, it is explained there just how our wave function is identical to the Fock expansion usually employed to define coherent states.

A curious fact is to be mentioned. The form of our special wave function coincides with that of a wave packet of Harmonic Oscillator eigen-functions studied in reference [8]. See also [9].

Accompanying potential function

We now set

$$y = x - \sqrt{2}\alpha_R, \qquad \text{--------(20)}$$

And in terms of y we obtain a Gaussian $\psi(y)$

$$\psi_\alpha(y) = \pi^{-\frac{1}{4}} e^{-y^2/2}. \qquad \text{---------- (21)}$$

Now, from Schroedinger's equation, given ψ, we have a potential function V (x) and an energy eigenvalues E given by

$$2(V(y)-E)=\frac{\psi''(y)}{\psi(y)}.$$

-----------(22)

We easily find

$$(1/2)\frac{\psi''(y)}{\psi(y)}=2y^2-1,$$ ---------(23)

Entailing

$$V(y)=y^2/2;\;\; y=x-\sqrt{2}\alpha_R,$$ ---------(24)

And, in the units used here,

$$E=1/2.$$ ----------(25)

Coherent states are eigenfunctions of special Hamiltonian H_S. H_S has an α_R-displaced quadratic potential function.

Conclusions

We see thus that coherent states are eigenfunctions of special Hamiltonian H_S. H_S has an α_R-displaced quadratic potential. The ground state of H_S has eigenvalue equal one-half (in the units used here). Note that the excited states of this potential (that involve Hermite polynomials) are not the excited coherent states advanced by Agarwal and Tara in [10], as the later involve Laguerre polynomials. See also [11]. The excited states of our potential are just the eigenfunctions of an α-displaced Harmonic Oscillator. To the best of the present author's knowledge, our excited states seem to not haven yet been empirically detected. At least, we were unable to find an appropriate reference.

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