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Review Article

Field Method for Solving Nonlinear Heat Conduction Equation[1]

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Abstract

We present a new method for integrating the nonlinear heat conduction problem. Using the field method, the basic partial differential equation is obtained for the non-linear heat conduction problem. The solution of non-linear heat conduction equation is derived when we put an actual condition into the basic partial differential equation. Two special cases in heat conduction problem are given to illustrate this method.

Keywords: Field Method; Heat Conduction Problem

Introduction

An important modern method in mechanical systems [1-5] for finding the integral, which is called field-method, is used to research the solution of a partial differential equation of second order that contains two variables. At first, we find a function that shows the relationship between these two variables, and then a more complicated partial differential equation of second order can be expressed by an ordinary differential equation. Then the field method in mechanical system is introduced for solving nonlinear heat conduction equation. The conclusion shows that field method in mechanical system can be fully used to find the solutions of a partial differential equation of second order, thus a new method for finding the solutions of second order equations is provided [6-10].

Using field method to solve heat conduction equation, it is necessary to make a field function variable be a function of other variables, at the same time for this function we construct a basic partial equation. If we can get a full integral, then the solution of heat conduction equation can be derived though algebra equation.

In this paper, we present field method to solve the nonlinear and nonhomogeneous heat conduction equation. We make a try that let x be function of t and this problem can be solved successfully. In the addition, variable functions will make different solutions.

Field method of integrating the nonconservative systems

The n degree-of-freedom mechanical system [1-3] with Lagrangian $L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$, subjected to a non-potential generalized force $Q_s^{'}(t,\boldsymbol{q},\dot{\boldsymbol{q}})$, where $\boldsymbol{q}=\{q_1,q_2,\cdots,q_n\}$, the equations of motion are written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s^{'} \quad (s=1,\cdots,n) \tag{1}$$

The equations of motion can be put in a form

$$\ddot{q}_s=f_s(t,q_k,\dot{q}_k) \quad (s,k=1,\cdots,n) \tag{2}$$

If $x_s=q_s$, $x_{n+s}=\dot{q}_s$ then Eq. (2) can be simplified to equations of order one

$$\dot{x}_s=x_{n+s}, \quad \dot{x}_{n+s}=f_s(x_s,x_{n+s},t) \quad (s=1,\cdots,n) \tag{3}$$

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Let a variable such as x_1 be a field function expressed in time t and other variables $x_2, \cdots, x_{2n}$.

i.e.

$$x_1 = u(t, x_a) \quad (a = 2, \cdots, 2n) \tag{4}$$

Differentiating the equation with respect to t and using Eq. (3), we obtain

$$\frac{\partial u}{\partial t} + \sum_{a=2}^{n} \frac{\partial u}{\partial x_a} x_{n+a} + \sum_{b=1}^{n} \frac{\partial u}{\partial x_{n+b}} f(t, u, x_A) - x_{n+1} = 0 \tag{5}$$

We call the linear partial differential equation (5) the basic partial differential equation.

Assume that the complete solution of Eq. (5) can be written in the form

$$x_1 = u(t, x_A, C_A) \quad (A = 2, \cdots, 2n). \tag{6}$$

Substituting Eq. (6) into (5) yields an identity. Let the initial value of x_0 be

$$x_a(0) = x_{a0} \quad (a = 1, \cdots, n)' \tag{7}$$

Substituting Eq. (7) into (6) where the constant C_1 is represented by x_{a0} and other constants $C_A(A=2,\cdots,2n)$, one can obtain

$$x_1 = u(t, x_A, x_{a0}, C_A) \tag{8}$$

It is easy to prove that the solution of Eq. (3) with initial values can be determined by formula (8), andthe following (2n-1) algebraic equations hold for any constants C_A

$$\frac{\partial u}{\partial C_A} = 0 \quad (A = 2, \cdots, 2n) \tag{9}$$

Field method of integrating the nonlinear heat conduction equation

We give the general form of nonlinear diffusion-convection equations of the form [9,10].

$$g(x)u_t = f(u)u_{xx} + h(u)u_x. \tag{10}$$

Apart from their intrinsic theoretical interest, the equation of type (10) is used to model a wide variety of phenomena in many fields, such as physics, engineering and chemistry etc.

We will introduce a nonlinear and nonhomogeneous heat conduction equation

$$g(x)u_t = f(u)u_{xx} + h(u)u_x \tag{11}$$

It is a partial differential equation that describes the flow of heat in a material in which the rate of heat flowis proportional to the temperature gradient. Make that

$$x = f_1(t) \tag{12}$$

And then we get the result

$$\frac{dt}{dx} = \left(\dot{f}_1\right)^{-1} \tag{13}$$

Assuming

$$\left(\dot{f}_1\right)^{-1} = g_1(t) \tag{14}$$

This means

$$u_x = u_t \frac{dt}{dx} = u_t \left(\dot{f}_1\right)^{-1} = u_t g_1(t) \tag{15}$$

With Eq. (15), Eq. (11) will have another form

$$g(x)u_t = f(u)\left[g_1(t)\dot{g}_1(t)u_t + g_1^2(t)u_{tt}\right] + h(u)g_1(t)u_t \tag{16}$$

Solving the system, we define

$$x_1 = u, x_2 = x, x_3 = \frac{\partial u}{\partial t}, x_4 = \frac{\partial x}{\partial t}, \tag{17}$$

Let x_3 be the field function which expressed by t and other variables, i.e.

$$x_3 = G(x_1, x_2, x_4, t), \tag{18}$$

Differentiating Eq. (18) with respect to t, the result is of the form,

$$\dot{x}_3 = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x_1}\dot{x}_1 + \frac{\partial G}{\partial x_2}\dot{x}_2 + \frac{\partial G}{\partial x_4}\dot{x}_4 \tag{19}$$

From Eq. (16), we get

$$x_3 = \frac{g(x) - f(u)g_1(t)\dot{g}_1(t) - h(u)g_1(t)}{f(u)g_1^2(t)}x_3 \quad (20)$$

With the transformation of Eq. (17) and (20), Eq. (19) will be of the form

$$A_1 - \frac{\partial G}{\partial x_1} = 0, \frac{\partial G}{\partial x_2} = 0, \frac{\partial G}{\partial t} = 0, \frac{\partial G}{\partial x_4} = 0 \quad (21)$$

Then G will be of the form (c is constant)

$$G = A_1 x_1 + c \quad (22)$$

Equation (22) equals to

$$\frac{\partial u}{\partial t} = A_1 u + c = \frac{g(x) - f(u)g_1(t)\dot{g}_1(t) - h(u)g_1(t)}{f(u)g_1^2(t)}u + c \quad (23)$$

The problem is reduced as an ODE which is simple to do.

Now we discuss two especial cases.

Case 1, with the condition of

$$g_1(t) = At, (A > 0) \quad (24)$$

$$f(u) = h(u) = B, \quad (25)$$

Where A and B are constants, then Eq. (24) can be of the form

$$\frac{\partial u}{\partial t} = \frac{g(x)}{BA^2t^2}u - \frac{1}{t^2}u - \frac{1}{At}u \quad (26)$$

The general solution can be written as:

$$u = -\frac{t}{A}e^{\frac{-g(x)+AB}{AB^2t}} \quad (27)$$

We can say that when $t \to 0$,at the same time, $\frac{-g(x)+BA}{A^2B} > 0$, i.e. with the additional case

,i.e. $x = e^{-t}$, (28)

$$f(u) = h(u) = Inu \quad (29)$$

For the special case

$$\frac{du}{u} = \left(\frac{g(x)}{e^{2t}Inu} - 1 - e^{-t}\right)dt, \quad (30)$$

$$Inu = -\frac{g(x)}{2Inu}e^{-2t} - t + e^{-t} + C, \text{ (C is constant)}, \quad (31)$$

Then there appears a function about $In\,u$ which can be written as

$$(Inu)^2 + Inu\left(t - e^{-t} - C\right) + \frac{g(x)}{2}e^{-2t} = 0 \quad (32)$$

Next, we talk about the most important function--- which can be given

$$\triangle = \left(t - e^{-t} - C\right)^2 - 2g(x)e^{-2t} \quad (33)$$

,i.e. $g(x) = \frac{\left(t - e^{-t} - C\right)^2}{2e^{-2t}}$, (34)

There exits $\triangle = 0$

$$u = e^{\frac{e^{-t} - t + C}{2}} \quad (35)$$

If $\triangle > 0$, there exists two values of u, they are

$$u_1 = e^{\frac{e^{-t} - t + C + \sqrt{\left(t - e^{-t} - C\right)^2 - 2g(x)e^{-2t}}}{2}}, u_2 = e^{\frac{e^{-t} - t + C - \sqrt{\left(t - e^{-t} - C\right)^2 - 2g(x)e^{-2t}}}{2}} \quad (36)$$

Due to C is arbitrary constant, so we can make C=0, with initial value

$$u(0,1) = \frac{1 + C - \sqrt{(1 + C)^2 - 2g(1)}}{2}, \quad (37)$$

We choose u_2.

For the generalized case

$$f(u) = h(u), \quad (38)$$

We can find out an implicit solution that can be written as

$$Inu = -\frac{g(x)}{2f(u)}e^{-2t} - t + e^{-t} + C, \quad (39)$$

For one case $t \to 0$, with the additional case

$$f(u) = Inu \text{。} \quad (40)$$

$$(Inu)^2 - (1+C)Inu + \frac{g(x)}{2} = 0 \quad (41)$$

$\Delta = 0$, i.e. (42)

We can get the result

$$u = e^{\frac{1+C}{2}}. \quad (43)$$

If $\triangle > 0$, we can make.

For another case

$$t \to \infty, u \to 0^- . \quad (44)$$

Conclusions

With an additional condition which has never been found, field method can solve the nonlinear and nonhomogeneous heat conduction equation successfully. We can extend this technique into another fields that one variable depends on the other variable.

The main contribution of this paper is, with an additional condition, that it presents now field integral method to solve the nonlinear nonhomogeneous heat conduction equation successfully. Further, we can also extend the method to another problem of mathematics and physics.

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