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Research Article

A Method to Solve One-dimensional Nonlinear Fractional Differential Equation Using B-Polynomials

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Abstract

In this article, the fractional Bhatti-Polynomial bases are applied to solve one-dimensional nonlinear fractional differential equations (NFDEs). We derive a semi-analytical solution from a matrix equation using an operational matrix which is constructed from the terms of the NFDE using Caputo's fractional derivative of fractional B-polynomials (B-polys). The results obtained using the prescribed method agree well with the analytical and numerical solutions presented by other authors. The legitimacy of this method is demonstrated by using it to calculate the approximate solutions to four NFDEs. The estimated solutions to the differential equations have also been compared with other known numerical and exact solutions. It is also noted that for solving the NFDEs, the present method provides a higher order of precision compared to the various finite difference methods. The current technique could be effortlessly extended to solving complex linear, nonlinear, partial, and fractional differential equations in multivariable problems.

Keywords: Fractional B-Polynomials; Fractional Differential Equations; Nonlinear Partial Fractional Differential Equation; FRACTIONAL B-polynomials in Multiple Variables

Introduction

The formulation of fractional calculus was started over 300 years ago. It can be traced back to Leibniz's letter to L'Hôpital, in which he first discussed the meaning of the one-half order derivative [1]. Although fractional calculus is as old as conventional calculus, it was not as widely used in engineering and science at its conception. Due to rapid advancements in the fields of mathematical physics, differential equations, interface chaos, and probability [2,3], as well as in other fields of science and engineering [4-9], fractional differential equations have become a subject of interest and a rapidly growing area of research. It is also used to describe a wide range of complex phenomena in different fields, such as anomalous diffusion, systems identification, wave propagation, continuous-time random walk dynamical systems, fractional electrical circuits, control theory, sub-diffusive systems, chaos synchronization, signal processing, viscoelasticity, fluid flow, and more [10-16]. Seismic analysis, viscoelastic materials, and viscous damping have all been successfully modeled in recent years using fractional differential equations (FDEs) [6,17-20]. The nonlinear oscillation of an earthquake can be modeled using fractional derivatives, and a fluid-dynamic traffic model using fractional derivatives can eliminate the deficiency caused by the assumption of continuous traffic flow [2,6,20]; as a result, developing robust methods for solving FDEs is essential. Many fractional-order differential equations have unknown exact solutions; thus, various numerical methods have been employed to provide approximate solutions. Unfortunately, each method has its own set of limitations, and while no single method can solve every problem, most techniques excel at solving specific problems.

There have been numerous approaches proposed to solve fractional differential equations, the widespread ones are the one with operational method [21,22], the Fourier transform method

[23], the iteration method [1], the iterative Laplace transform method (ILTM) [17], the Bernoulli wavelet method [24], the Spectral method [25], and the Laplace transform method [6,20]. The approaches vary in their strengths and weaknesses, but from the variety comes a new factor in solving FDEs, namely computation time, with some methods requiring a significant amount of computational time to accomplish solutions to fractional-order differential equations.

In this article, we present a novel technique known as the modified fractional Bhatti Polynomials method. With this method, we have successfully solved linear and nonlinear differential equations; see references [26-29]. The method is effective, and the results obtained thus far are encouraging and reliable. Four examples are provided in this article to explain the dependability and efficacy of the method. The results of the method are also compared with existing techniques, and excellent agreement has been found between the results; in all the cases, the present semi-analytical results are superior in accuracy.

Caputo's Fractional differential operator

The fractional-order derivative of Caputo is explained as follows [6],

$$D^{\gamma}f(x) = J^{m-\gamma}D^{m}f(x) = \frac{1}{\Gamma(m-\beta)}\int_0^x (x-t)^{m-\gamma-1}f^{(m)}(t)dt, \quad (1)$$

for $m-1<\gamma\le m$, continous where $m\in N, x>0, f\in C_{-1}^{m}$,

D^{γ} is Caputo's fractional derivative operator. The Caputo's fractional derivative for any constant, C, is zero such that: $D^{\gamma}C = 0$, and the fractional derivative $D_x^{\gamma}x^{\alpha}$ is given by:

$$D^{\gamma}x^{\alpha} = \begin{cases} 0 \text{ for } \alpha\in N_0 \text{ and } \alpha<[\gamma] \\ \dfrac{\Gamma(\alpha+1)\,x^{\alpha-\gamma}}{\Gamma(\alpha+1-\gamma)} \text{ otherwise.}\end{cases} \quad (2)$$

The fractional order of the function is denoted by α, and the fractional order of the derivative is given by . The unknown function is expanded in terms of the generalized fractional-order B-polynomials $B_{i,n}(\alpha,x)$, which can be regarded as an approximate solution to the one-dimensional NFD equation:

$$y(x) = \sum_{i=0}^{n} b_i\, B_{i,n}(\alpha,x) + f(x). \quad (3)$$

In variable x, $B_{i,n}(\alpha,x)$ is ith fractional-order B-poly with α as a fractional-order parameter and is the initial condition imposed on the solution. The expansion coefficients represent the expansion coefficients that are determined in the Galerkin scheme of minimization in Eq. (3). Fractional calculus of differentiation can be accomplished using Caputo's derivative as a linear operator:

$$D_x^{\gamma}\left(\sum_{i=0}^{n} b_i\, B_{i,n}(\alpha,x) + f(x)\right) = \sum_{i=0}^{n} b_i\left(D_x^{\gamma}\left(B_{i,n}(\alpha,x)\right)\right) + D_x^{\gamma}(f(x)). \quad (4)$$

The generalized fractional-order B-poly basis and some of its properties that may be useful in determining a solution to the nonlinear fractional-order differential equation are briefly discussed in the following section.

Fractional-order B-Poly basis

The generalization of fractional B-polys in terms of single variable x over the interval [0, R] is defined in [27,28],

$$B_{i,n}(\alpha,x) = \sum_{k=0}^{n} \beta_{i,k}\left(\frac{x}{R}\right)^{\alpha k}. \quad (5)$$

The fractional-order parameter α represents the fractional B-polys and the Eq. (5) provides an (n+1) fractional-order B-polynomial basis set. In Eq. (5), the factor is defined as:

$$\beta_{i,k} = (-1)^{i-k}\binom{n}{k}\binom{k}{i}, \quad (6)$$

And the binomial coefficient is defined as: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Using a simple symbolic code prewritten with any value of n supported over an interval [0, R], it is possible to produce a fractional B-poly basis set. The boundary conditions are typically associated with the first and the last polynomials in the basis set. For example, the fractional basis set for n =3 and $\alpha=\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{3}{4}$, and $\frac{4}{5}$, are given for various values of fractional order in table 1.

Method for approximating solutions of one-dimensional NFDEs

Using the Galerkin method [28] and the generalized fractional-order B-poly basis set, we exploit a method to seek practical solutions to nonlinear fractional-order differential equations (NFDEs). Using the recently developed method [26-32], we transform the fractional-order NFDE into an operational matrix with initial and boundary conditions imposed on it. To construct the operational matrix, we substitute Eq. (3) into the given NFDE,

Table 1: When $\gamma = \alpha = \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}$, and $\frac{4}{5}$, the corresponding basis set and their derivative for n = 3 are given in this table. The Gamma function of fractional order is represented by Γ[a/b].

α	γ	n	Basis Set (n+1)	Caputo's Derivative of Basis set (Equation (4))
$\frac{1}{2}$	$\frac{1}{2}$	3	$\{1 - 3\sqrt{x} + 3x - x^{3/2}, 3\sqrt{x} - 6x + 3x^{3/2}, 3x - 3x^{3/2}, x^{3/2}\}$	$\{-\frac{3\sqrt{\pi}}{2} + \frac{6\sqrt{x}}{\sqrt{\pi}} - \frac{3\sqrt{\pi}x}{4}, \frac{3\sqrt{\pi}}{2} - \frac{12\sqrt{x}}{\sqrt{\pi}} + \frac{9\sqrt{\pi}x}{4}, \frac{6\sqrt{x}}{\sqrt{\pi}} - \frac{9\sqrt{\pi}x}{4}, \frac{3\sqrt{\pi}x}{4}\}$
$\frac{3}{5}$	$\frac{3}{5}$	3	$\{1 - 3x^{3/5} + 3x^{6/5} - x^{9/5}, 3x^{3/5} - 6x^{6/5} + 3x^{9/5}, 3x^{6/5} - 3x^{9/5}, x^{9/5}\}$	$\{-3\Gamma[\frac{8}{5}] + \frac{3x^{3/5}\Gamma[\frac{11}{5}]}{\Gamma[\frac{8}{5}]} - \frac{x^{6/5}\Gamma[\frac{14}{5}]}{\Gamma[\frac{11}{5}]}, 3\Gamma[\frac{8}{5}] - \frac{6x^{3/5}\Gamma[\frac{11}{5}]}{\Gamma[\frac{8}{5}]} + \frac{3x^{6/5}\Gamma[\frac{14}{5}]}{\Gamma[\frac{11}{5}]}, \frac{3x^{3/5}\Gamma[\frac{11}{5}]}{\Gamma[\frac{8}{5}]} - \frac{3x^{6/5}\Gamma[\frac{14}{5}]}{\Gamma[\frac{11}{5}]}, \frac{x^{6/5}\Gamma[\frac{14}{5}]}{\Gamma[\frac{11}{5}]}\}$
$\frac{2}{3}$	$\frac{2}{3}$	3	$\{1 - 3x^{2/3} + 3x^{4/3} - x^2, 3x^{2/3} - 6x^{4/3} + 3x^2, 3x^{4/3} - 3x^2, x^2\}$	$\{-3\Gamma[\frac{5}{3}] - \frac{2x^{4/3}}{\Gamma[\frac{7}{3}]} + \frac{3x^{2/3}\Gamma[\frac{7}{3}]}{\Gamma[\frac{5}{3}]}, 3\Gamma[\frac{5}{3}] + \frac{6x^{4/3}}{\Gamma[\frac{7}{3}]} - \frac{6x^{2/3}\Gamma[\frac{7}{3}]}{\Gamma[\frac{5}{3}]}, -\frac{6x^{4/3}}{\Gamma[\frac{7}{3}]} + \frac{3x^{2/3}\Gamma[\frac{7}{3}]}{\Gamma[\frac{5}{3}]}, \frac{2x^{4/3}}{\Gamma[\frac{7}{3}]}\}$
$\frac{3}{4}$	$\frac{3}{4}$	3	$\{1 - 3x^{3/4} + 3x^{3/2} - x^{9/4}, 3x^{3/4} - 6x^{3/2} + 3x^{9/4}, 3x^{3/2} - 3x^{9/4}, x^{9/4}\}$	$\{\frac{9\sqrt{\pi}x^{3/4}}{4\Gamma[\frac{7}{4}]} - 3\Gamma[\frac{7}{4}] - \frac{4x^{3/2}\Gamma[\frac{13}{4}]}{3\sqrt{\pi}}, -\frac{9\sqrt{\pi}x^{3/4}}{2\Gamma[\frac{7}{4}]} + 3\Gamma[\frac{7}{4}] + \frac{4x^{3/2}\Gamma[\frac{13}{4}]}{\sqrt{\pi}}, \frac{9\sqrt{\pi}x^{3/4}}{4\Gamma[\frac{7}{4}]} - \frac{4x^{3/2}\Gamma[\frac{13}{4}]}{\sqrt{\pi}}, \frac{4x^{3/2}\Gamma[\frac{13}{4}]}{3\sqrt{\pi}}\}$
$\frac{4}{5}$	$\frac{4}{5}$	3	$\{1 - 3x^{4/5} + 3x^{8/5} - x^{12/5}, 3x^{4/5} - 6x^{8/5} + 3x^{12/5}, 3x^{8/5} - 3x^{12/5}, x^{12/5}\}$	$\{-3\Gamma[\frac{9}{5}] + \frac{3x^{4/5}\Gamma[\frac{13}{5}]}{\Gamma[\frac{9}{5}]} - \frac{x^{8/5}\Gamma[\frac{17}{5}]}{\Gamma[\frac{13}{5}]}, 3\Gamma[\frac{9}{5}] - \frac{6x^{4/5}\Gamma[\frac{13}{5}]}{\Gamma[\frac{9}{5}]} + \frac{3x^{8/5}\Gamma[\frac{17}{5}]}{\Gamma[\frac{13}{5}]}, \frac{3x^{4/5}\Gamma[\frac{13}{5}]}{\Gamma[\frac{9}{5}]} - \frac{3x^{8/5}\Gamma[\frac{17}{5}]}{\Gamma[\frac{13}{5}]}, \frac{x^{8/5}\Gamma[\frac{17}{5}]}{\Gamma[\frac{13}{5}]}\}$

then Caputo's derivative operator is applied to the basis set used in the expansion in each term of the NFDE, and both sides of the NFDE are multiplied by the elements of the fractional B-poly basis set, . Finally, the integrations are carried out using the symbolic program Mathematica [33,34] over the closed interval [0, R] into interaction matrix. For example, the integration matrix over the closed interval of the two fractional-order B-polys is given in the closed symbolic formula:

$$m_{i,j} = \left(B_{i,n}(\alpha,x), B_{j,n}(\alpha,x)\right) = \sum_{k=i}^{n} \beta_{i,k} \left(\frac{X}{R}\right)^{\alpha k} \sum_{l=j}^{n} \alpha_{j,k} \left(\frac{X}{R}\right)^{\alpha l} \frac{R}{(k+l)\alpha}. \quad (7)$$

The Caputo's derivative defined in Eq. (2) is applied to the fractional B-poly basis set, leading to the following closed results:

$$D_x^{\gamma}\left(B_{i,n}(\alpha,x)\right) = \sum_{k=i}^{n} \alpha_{i,k} D_x^{\gamma}\left(\frac{X}{R}\right)^{\alpha k} = \sum_{k=i}^{n} \frac{\beta_{i,k}}{R^{\alpha k}} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+1-\gamma)} x^{\alpha k-\gamma},$$

$$d_{i,l}^{(\gamma)}(x) = \left(D_x^{\gamma}B_{i,n}(\alpha,x), B_{l,n}(\alpha,x)\right) = \langle D_x^{\gamma}B_{i,n}(\alpha,x) | B_{l,n}(\alpha,x)\rangle$$
$$= \sum_{k=i,l=j}^{n} \beta_{i,k}\,\beta_{l,k} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+1-\gamma)} \frac{R^{1-\gamma}}{((k+l)\alpha+1-\gamma)}, \quad (8)$$

And the integrals of some arbitrary function are given by,

$$W_m = \int_0^R f(x)\, B_m(\alpha,x)dx. \quad (9)$$

With the help of the above analytic formulas, Eqs. (7-9), the operational matrix is formulated. The inverse of the operational matrix is required to find out the unknown coefficients of the linear combination in Eq. (3). In the next section, we will apply our method and demonstrate how one can obtain a desirable solution to the nonlinear fractional-order differential equation. The method will be applied to four examples to demonstrate that it works appropriately for approximating the solutions with greater accuracy. We will also explain how the inverse of the operational matrix is calculated using the symbolic program Mathematica 13.0 [33,34]. Plots of the approximate and exact solutions will be presented for the purpose of making comparisons. Also, the absolute error analysis of the fourth example will be elaborated to show that by including larger basis set of the fractional B-polys and increasing the number of iterations used to solve NFDEs, the accuracy of the solution is enhanced considerably. In the following sections, for the sake of simplicity, we will drop the subscript from the fractional B-poly basis, so that .

Example 1: Consider a one-dimensional nonlinear fractional-order differential equation,

$$D^{\gamma}y(x) - y(x) + 2y^{2}(x) = 0, \qquad (10)$$

Where the value of $0 \leq \gamma \leq 1$ with the initial condition $y(0) = 1/3$.

The exact solution of this equation when $\gamma = 1$ known to be:

$y_{exact}(x) = 1/(2 + e^{-x})$. Using fractional B-poly basis, a solution may be approximated as $y_{app}(x) = \sum_{i=0}^{n} b_i B_i(\alpha, x) + y_0$ with the initial condition, $y_0 = 1/3$.

.By substituting the approximate solution into the Eq. (10), we obtain,

$$\frac{d^{\gamma}}{dx^{\gamma}}\left(\sum_{i=0}^{n} b_i B_i(\alpha,x) + y_0\right) - \left(\sum_{i=0}^{n} b_i B_i(\alpha,x) + y_0\right) + 2\left(\sum_{j=0}^{n} b_j B_j(\alpha,x) + y_0\right)\left(\sum_{i=0}^{n} b_i B_i(\alpha,x) + y_0\right) = 0. \qquad (11)$$

We evaluate Eq. (11) by computing the Caputo fractional derivative, multiplying both sides of the equation by the elements of the fractional B-polys basis set, $B_l(\alpha, x)$, , and carrying out the integration over the interval [0, R],

$$\sum_{i}^{n} b_i \left[\langle D^{\gamma}B_i(\alpha,x)|B_l(\alpha,x)\rangle - \langle B_i(\alpha,x)|B_l(\alpha,x)\rangle + 4\langle y_0 B_i(\alpha,x)|B_l(\alpha,x)\rangle + 2\sum_{j}^{n} b_j \langle B_i(\alpha,x)\, B_j(\alpha,x)|B_l(\alpha,x)\rangle\right] = \langle (y_0 - 2y_0^2)\, B_l(\alpha,x)|B_l(\alpha,x)\rangle. \qquad (12)$$

The above equation may be rewritten in the matrix form,

$$\Rightarrow B\,[A - C + D + E] = W, \qquad (13)$$

Where coefficients of the 4th term, E, exhibit nonlinearity via its coefficients. The matrices of Eq. (13) are given below:

$$\begin{aligned} A &= \langle D^{\gamma}B_i(\alpha,x)|B_l(\alpha,x)\rangle = \int_0^R D^{\gamma}(B_i(\alpha,x))B_l(\alpha,x)\,dx, \\ C &= \langle B_i(\alpha,x)|B_l(\alpha,x)\rangle = \int_0^R B_i(\alpha,x)B_l(\alpha,x)\,dx, \\ D &= 4\langle y_0 B_i(\alpha,x)|B_l(\alpha,x)\rangle = 4\int_0^R y_0\, B_i(\alpha,x)B_l(\alpha,x)\,dx, \\ E &= 2\sum_{j}^{n} b_j g_{ijl} = 2\sum_{j}^{n} b_j\langle B_i(\alpha,x)\,B_j(\alpha,x)|B_l(\alpha,x)\rangle = 2\sum_{j}^{n} b_j \int_0^R B_i(\alpha,x)B_j(\alpha,x)B_l(\alpha,x)\,dx, \\ W &= \langle (y_0 - 2y_0^{2})B_l(\alpha,x)|B_l(\alpha,x)\rangle = \int_0^R (y_0 - 2y_0^2)B_l(\alpha,x)B_l(\alpha,x)\,dx. \end{aligned} \qquad (14)$$

For the initial estimates of the coefficients, we ignore nonlinear terms in equation (13) to calculate matrix B for unknown coefficients b and the equation is solved to obtain initial guess,

$$B\,[A - C + D] = W. \qquad (15)$$

The new estimate for the value of matrixcan be calculated using equation (13) and using initial estimate from equation (15). After a few more iterations, we revise our estimated solution for comparison. We also solve the nonlinear fractional differential equation for different fractional values of γ by repeating the same procedure. The graphs are plotted for various values of γ alongside the exact solution to observe the deviation from the exact solution for $\gamma = 1$ integral value, figure 2.

Figure 1: The approximate solution f(x) and the precise solution (sol) are shown in figure 1 for the case $\gamma=1$ in equation (11), demonstrating that the both solutions overlap. In the picture on the right, the absolute error between the exact and approximate solutions is displayed. The absolute error is of the order of 10^(-9).

Figure 1 shows that when γ is equal to 1, the graphs of numerical convergent $f(x)$ solution and exact (sol) solution are shown on the left side. The order of absolute error between estimated $f(x)$ and accurate (sol) solutions is given on the right side which is of the order 10^{-9} of. Higher accuracy can be accomplished if the number of fractional B-poly set is increased. In the references [35,36], the absolute error is 10^{-3} using their numerical technique. As a result, our method produces a highly accurate solution.

Figure 2: Various fractional values of γ=1,4/5,3/4,2/3,1/2 are used in equation (11) and the plots of approximate solutions are presented in this figure. It is noted that all the graphs approximately intersect at one point (x ≅ 1).

Example 2: Let us consider the following nonlinear fractional differential equation:

$$D^{\gamma}y(x) = 2y(x) - y^2(x) + 1, \tag{16}$$

Where the value of $0 \le \gamma \le 1$ and the boundary conditions $y(0) = y(R) = 0$ are imposed on Eq. (16). The exact solution of this equation when $\gamma = 1$ known: $y_{exact}(x) = 1 + \sqrt{2}\tanh(\sqrt{2}x + \frac{1}{2}\log(\frac{\sqrt{2}-1}{\sqrt{2}+1}))$.

Using the fractional B-poly basis, a solution may be approximated as $y_{app}(x) = \sum_i^n b_i B_i(\alpha, x) + y_0$, with the initial condition, $y_0 = 0$. By plugging this approximate solution into Eq. (16), we get the following expression:

$$\frac{d^{\gamma}}{dx^{\gamma}}\left(\sum_i^n b_i B_i(\alpha,x) + y_0\right) = 2\left(\sum_i^n b_i B_i(\alpha,x) + y_0\right) - \left(\sum_i^n b_i B_i(\alpha,x) + y_0\right)\left(\sum_j^n b_j B_j(\alpha,x) + y_0\right) + 1. \tag{17}$$

We evaluate Eq. (17) by applying the Caputo fractional derivative on the first term, multiplying both sides of the equation by the elements of the fractional B-polys basis set, $B_l(\alpha, x)$, and then carrying out the integration over the interval [0, R] on both sides. We get,

$$\sum_i^n b_i \left[\langle D^{\gamma}B_i(\alpha,x)|B_l(\alpha,x)\rangle - 2\langle B_i(\alpha,x)|B_l(\alpha,x)\rangle + 2\langle y_0\, B_i(\alpha,x)|B_l(\alpha,x)\rangle + \sum_j^n b_j \langle B_i(\alpha,x)B_j(\alpha,x)|B_l(\alpha,x)\rangle\right] = \langle (2\,y_0 - y_0^2 + 1)B_l(\alpha,x)|B_l(\alpha,x)\rangle. \tag{18}$$

The above equation can be represented in the matrix form:

$$B\,[A - C + D + E] = W, \tag{19}$$

With the elements of each matrix in Eq. (19) are given,

$$\begin{aligned}
A &= \langle D^{\gamma}B_i(\alpha,x)|B_l(\alpha,x)\rangle = \int_0^R D^{\gamma}(B_i(\alpha,x))B_l(\alpha,x)\,dx,\\
C &= 2\langle B_i(\alpha,x)|B_l(\alpha,x)\rangle = 2\int_0^R B_i(\alpha,x)B_l(\alpha,x)\,dx,\\
D &= 2\langle y_0\, B_i(\alpha,x)|B_l(\alpha,x)\rangle = 2\int_0^R y_0\, B_i(\alpha,x)B_l(\alpha,x)\,dx,\\
E &= \sum_j^n b_j\, g_{ijl} = b_j\langle B_i(\alpha,x)\,B_j(\alpha,x)|B_l(\alpha,x)\rangle = b_j\int_0^R B_i(\alpha,x)B_j(\alpha,x)B_l(\alpha,x)\,dx,\\
W &= \langle (2\,y_0 - y_0^2 + 1)B_l(\alpha,x)|B_l(\alpha,x)\rangle = \int_0^R (2y_0 - y_0^2 + 1)\,B_l(\alpha,x)B_l(\alpha,x)\,dx.
\end{aligned} \tag{20}$$

For the initial guess of the coefficients, we can solve Eq. (21) by neglecting the nonlinear term. We calculate matrix elements of column matrix as an initial guess by solving the equation,

$$B\,[A - C + D] = W. \tag{21}$$

We substitute the elements of into nonlinear Eq. (19) to obtain the revised estimate of the unknown coefficients. The process of iteration is repeated until a convergent solution is found. We have used the same procedure to solve the nonlinear fractional differential equation for various fractional values of γ. The graphs of the solutions for several values of are shown in figure 3. In each, the accuracy was desirable as shown in figure 4.

Figure 3: A 1-D graph of the approximate solution f(x) and exact (sol) solution is displayed on the left side, demonstrating how well the two solutions overlap for γ=1. In the picture on the right side, the absolute error between the precise and approximate solutions is displayed. The graph shows that the nonlinear solutions is converged and has high accuracy of the solution in the order of 10^{-10}.

Higher-order accuracy can be accomplished if the number of fractional B-Polys and iterations are increased. In the other references [35,36], the absolute error is of the order of 10^{-3} and 10^{-6} but our method accomplishes higher-order computational accuracy. Figure 4 depicts solutions of example 2 with various fractional order $\gamma = 1, \frac{4}{5}, \frac{3}{4}, \frac{2}{3}, \frac{1}{2}, \frac{3}{5}$.

Tables 2, 3, and 4 show that our method produces excellent results when compared with approximate other methods for various fractional values [35-37] shown in these tables. The accuracy of our method, as pointed out in the earlier works [29,31,38], was dependent on the number of basis sets and number of iterations to achieve converged solution. We accept convergent values after a few iterations are performed. Tables 2, 3 and 4 show comparison of present results and other results which are obtained from using various finite difference methods.

Figure 4: Nonlinear Equation (16) is solved using fractional B-polys to produce the plots of various approximate solutions shown in the figure. To produce these graphs, various fractional-order values for γ=1,4/5,3/4,2/3,1/2,3/5 are used.

Table 2: For fractional order derivative $\gamma = 1$, the comparison of the fractional B-poly approach to other methods [35-37] is shown, where N denotes the number of B-polys in a basis set.

$\gamma = 1$								
	Our Results							
x	N = 8	N = 10	N = 12	N = 15	y_{haar}[35]	y_{hpm}[37]	y_{cwm}[36]	y_{exact}[37]
0.0	0	0	0	0	0	0	0	0
0.1	0.110284	0.110295	0.110295	0.110295	0.110295	0.110294	0.110311	0.110295
0.2	0.241965	0.241977	0.241977	0.241977	0.241977	0.241965	0.241995	0.241977
0.3	0.395087	0.395105	0.395105	0.395105	0.395105	0.395106	0.395123	0.395105
0.4	0.567796	0.567813	0.567812	0.567812	0.567813	0.568115	0.567829	0.567812
0.5	0.755999	0.756015	0.756014	0.756014	0.756015	0.757564	0.756029	0.756014
0.6	0.953546	0.953567	0.953566	0.953566	0.953567	0.958259	0.953576	0.953566
0.7	1.152930	1.152950	1.152950	1.152950	1.152949	1.163459	1.152955	1.152949
0.8	1.346350	1.346360	1.346360	1.346360	1.346364	1.365240	1.346365	1.346364
0.9	1.526890	1.526910	1.526910	1.526910	1.526911	1.554960	1.526909	1.526911
1.0	1.689480	1.689500	1.689500	1.689500	1.689499	1.723810	1.689494	1.689498

Table 3: For fractional order derivatives $\gamma = 0.5$ and 0.75, the comparison of the current fractional B-poly approach to other methods [36,37] is presented, where N denotes the number of basis sets.

	$\gamma = 0.5$					$\gamma = 0.75$				
	Our Results					Our Results				
x	N = 6	N = 7	N = 8	y_{hpm}[37]	y_{cwm}[36]	N = 6	N = 7	N = 8	y_{hpm}[37]	y_{cwm}[36]
0.0	0	0	0	0	0	0	0	0	0	0
0.1	0.59369	0.59386	0.59308	0.32173	0.59276	0.24623	0.24552	0.24534	0.21687	0.31073
0.2	0.93264	0.93341	0.93306	0.62967	0.93318	0.47613	0.47536	0.47490	0.42889	0.58431
0.3	1.17353	1.17382	1.17402	0.94094	1.17398	0.71154	0.71040	0.70991	0.65461	0.82217
0.4	1.34705	1.34697	1.34684	1.25074	1.34665	0.93952	0.93876	0.93840	0.89140	1.02497
0.5	1.47428	1.47433	1.47388	1.54944	1.47389	1.14929	1.14928	1.14889	1.13276	1.19862
0.6	1.57025	1.57069	1.57044	1.82546	1.57057	1.33421	1.33475	1.33421	1.37024	1.34915
0.7	1.64538	1.64607	1.64622	2.06652	1.64620	1.49157	1.49239	1.49187	1.59428	1.48145
0.8	1.70635	1.7069	1.70702	2.26063	1.70688	1.62213	1.62337	1.62304	1.79488	1.59924
0.9	1.75669	1.75702	1.75659	2.39684	1.75664	1.72935	1.7314	1.73118	1.96223	1.70530
1.0	1.79724	1.79826	1.79839	2.46600	1.79822	1.8188	1.82087	1.82093	2.08738	1.80176

Table 4: For fractional order derivatives $\gamma = 0.6$ and 0.8, the comparison of the current fractional B-poly approach to other methods [35] is shown, where N denotes the number of B-polys in the basis set.

γ = .6					γ = .8				
x	**N = 6**	**N = 7**	**N = 8**	**y_{haar} [35]**	**x**	**N = 6**	**N = 7**	**N = 8**	**y_{haar} [35]**
0.0	0	0	0	0	0.0	0	0	0	0
0.1	0.41362	0.41353	0.41311	0.42697	0.1	0.20899	0.20796	0.20788	0.211942
0.2	0.72356	0.72304	0.72221	0.73068	0.2	0.41430	0.41339	0.41323	0.41000
0.3	0.98636	0.98574	0.98557	0.99053	0.3	0.63305	0.63155	0.63136	0.635149
0.4	1.2006	1.20001	1.20004	1.19393	0.4	0.85450	0.85299	0.85286	0.85002
0.5	1.37101	1.37010	1.36995	1.37789	0.5	1.06752	1.06644	1.06633	1.06670
0.6	1.50529	1.50339	1.50342	1.48518	0.6	1.26334	1.26253	1.26239	1.27900
0.7	1.61173	1.60808	1.60886	1.62822	0.7	1.4363	1.43539	1.43525	1.43387
0.8	1.6978	1.69153	1.69314	1.85574	0.8	1.58398	1.58298	1.5829	1.58714
0.9	1.7693	1.75913	1.76125	2.16946	0.9	1.707	1.70639	1.70632	1.70696
1.0	1.8298	1.81342	1.81719	2.73017	1.0	1.80874	1.80814	1.80808	1.80913

Example 3: Consider another nonlinear fractional differential equation of the form,

$$D^{\gamma}y(x) = -y^2(x) + 1, \tag{22}$$

Where the value of $0 \leq \gamma \leq 1$ and the boundary conditions $y(0) = y(R) = 0$ are given. The exact solution of this equation (22) when is $\gamma = 1$ is $y_{exact}(x) = (e^{2x} - 1)/(e^{2x} + 1)$. An approximate solution may be written as $y_{app}(x) = \sum_i^n b_i B_i(\alpha, x) + y_0$, where initial condition, $y_0 = 0$ is given. Plugging in the approximate solution into the Eq. (22), we get,

$$\frac{d^{\gamma}}{dx^{\gamma}}\left(\sum_i^n b_i B_i(\alpha, x) + y_0\right) = -\left(\sum_i^n b_i B_i(\alpha, x) + y_0\right)\left(\sum_j^n b_j B_j(\alpha, x) + y_0\right) + 1. \tag{23}$$

Evaluating Eq. (23) by computing the Caputo's fractional derivative in the first term, multiplying both sides of the equation by the elements of the fractional B- basis set, $B_l(\alpha, x)$, and integrating over the interval [0, R] on both sides of the Eq. (23), we obtain,

$$\sum_i^n b_i \left[\langle D^{\gamma}B_i(\alpha, x)|B_l(\alpha, x)\rangle + 2\langle y_0 B_i(\alpha, x)|B_l(\alpha, x)\rangle + \sum_j^n b_j \langle B_i(\alpha, x)\, B_j(\alpha, x)|B_l(\alpha, x)\rangle\right] = \langle(1 - y_0^2)B_i(\alpha, x)|B_l(\alpha, x)\rangle. \tag{24}$$

The following is a representation of the above equation in matrix form,

$$B\,[A + D + E] = W, \tag{25}$$

Where elements of each matrix are given as follows:

$$A = \langle D^{\gamma}B_i(\alpha, x)|B_l(\alpha, x)\rangle = \int_0^R D^{\gamma}(B_i(\alpha, x))B_l(\alpha, x)\, dx, \tag{26}$$

$$D = 2\langle y_0 B_i(\alpha, x)|B_l(\alpha, x)\rangle = 2\int_0^R y_0 B_i(\alpha, x)B_l(\alpha, x)\, dx,$$

$$E = \sum_j^n b_j\, g_{jkl} = b_j\langle B_i(\alpha, x)\, B_j(\alpha, x)|B_l(\alpha, x)\rangle = b_j\int_0^R B_i(\alpha, x)B_j(\alpha, x)B_l(\alpha, x)\, dx.$$

$$W = \langle(1 - y_0^2)B_i(\alpha, x)|B_l(\alpha, x)\rangle = \int_0^R (1 - y_0^2)B_i(\alpha, x)B_l(\alpha, x)\, dx.$$

By ignoring the nonlinear term, we may solve the equation for initial guess such that

$$B\,[A + D] = W. \tag{27}$$

The initial guess of Eq. (27) is substituted for the nonlinear term to obtain a new guess for matrix B by solving equation (25). This process is repeated until convergent values of the coefficients of matrix B are obtained. We have solved the nonlinear fractional differential equation Eq. (25) for different fractional-order values of using the same procedure. The graphs are also plotted of various solutions for values of in the figure 5 and 6. The order of absolute error between estimated $f(x)$ and accurate (sol) solutions on the right side is 10^{-9}. Higher accuracy can be accomplished if the number of fractional B-Poly basis set and iterations are increased. In the reference [35], the absolute error is of the order 10^{-3} for the same problem. Our method performs exceptionally well in terms of computation accuracy.

Figure 5: For fractional derivative γ=1 in equation (22), a 1-D graphic of our approximate solution, f(x), and the precise solution, (sol), are displayed on the left side showing that the two solutions overlap. The image on the right shows a plot of the absolute error between the approximate and precise solutions on the order of 10^{-9}.

Figure 6: The fractional differential equation (22), which considers various fractional orders γ=1,4/5,3/4,2/3,1/2 , is shown in the picture along with a plot of several approximations. All the graphs intersect at one point (x≅1).

Tables 5, 6, and 7 show that fractiona B-poly method produces excellent results when compared to other numerical methods for various fractional-order values [35,37]. The accuracy of our method [29,31] was determined by the number of B-polys used in the basis set and the number of iterations used. The convergent values are obtained after a few iterations.

Table 5: For fractional order $\gamma = 1$, the comparison of the fractional B-poly approach to other methods [35,37] has been presented, where N denotes the number of B-polys in the basis set.

$\gamma = 1$							
	Our Results						
x	N = 6	N = 7	N = 8	N = 9	y_{haar}[35]	y_{hpm}[37]	y_{exact}[37]
0.0	0	0	0	0	0	0	0
0.1	0.099652	0.099669	0.099668	0.099668	0.099668	0.099668	0.099668
0.2	0.197366	0.197376	0.197376	0.197375	0.197375	0.197375	0.197375
0.3	0.291303	0.291314	0.291313	0.291313	0.291313	0.291312	0.291313
0.4	0.379935	0.379950	0.379949	0.379949	0.379949	0.379944	0.379949
0.5	0.462105	0.462118	0.462117	0.462117	0.462117	0.462078	0.462117
0.6	0.537042	0.537050	0.537050	0.537050	0.537050	0.536857	0.537050
0.7	0.604362	0.604369	0.604368	0.604368	0.604368	0.603631	0.604368
0.8	0.664028	0.664038	0.664037	0.664037	0.664037	0.661706	0.664037
0.9	0.716290	0.716298	0.716298	0.716298	0.716298	0.709919	0.716298
1.0	0.761589	0.761595	0.761594	0.761594	0.761594	0.746032	0.761594

Table 6: For fractional order $\gamma = 0.75$,, the comparison of the fractional B-poly method to other numerical methods [37] is presented, where N denotes the number of B-polys in the basis set.

γ = 0.5					
	Our Results				
x	N = 6	N = 7	N = 8	N = 9	y_{hpm} [37]
0.0	0	0	0	0	0
0.1	0.190088	0.190101	0.190101	0.190097	0.184795

0.2	0.309960	0.309973	0.309976	0.309960	0.313795
0.3	0.404581	0.404612	0.404614	0.404612	0.414562
0.4	0.481611	0.481632	0.481632	0.481630	0.492889
0.5	0.545088	0.545090	0.545090	0.545081	0.462117
0.6	0.597783	0.597781	0.597783	0.597777	0.597393
0.7	0.641807	0.641819	0.641820	0.641821	0.631772
0.8	0.678832	0.678850	0.678850	0.678847	0.660412
0.9	0.710173	0.710175	0.710175	0.710170	0.687960
1.0	0.736827	0.736837	0.736837	0.736834	0.718260

Table 7: For fractional order $\gamma = 0.5$, the comparison of the fractional B-poly method to other numerical methods [35,37] is presented, where N denotes the number of B-polys in the basis set.

	Our Results					
x	N = 6	N = 7	N = 8	N = 9	y_{haar} [35]	y_{hpm} [37]
0.0	0	0	0	0	0	0
0.1	0.330098	0.330097	0.330107	0.330096	0.324691	0.273875
0.2	0.436815	0.436838	0.43684	0.436846	0.432214	0.454125
0.3	0.504894	0.504896	0.504889	0.504888	0.504115	0.573932
0.4	0.553794	0.553780	0.553781	0.553775	0.553825	0.644422
0.5	0.591194	0.591187	0.591194	0.591194	0.590729	0.674137
0.6	0.621003	0.621012	0.621015	0.621019	0.622213	0.671987
0.7	0.645476	0.645491	0.645487	0.645487	0.643153	0.648003
0.8	0.666022	0.666024	0.666021	0.666016	0.667030	0.613306
0.9	0.683566	0.683553	0.683558	0.683560	0.680422	0.579641
1.0	0.698737	0.698755	0.698748	0.698743	0.695251	0.558557

Example 4: Consider a final example of the nonlinear fractional differential equation,

$$D^{\gamma}y(x) - y(x) + y^2(x) + y'^2(x) - 1 = 0, \qquad (28)$$

Where the value of 0 ≤γ≤1 and the boundary conditions are specified as y(0) = y(R) = 0. The exact solution of the fractional differential equation for integral order γ = 2 is $y_{exact}(x) = 1+\cos(x)$. To determine approximate solution, we consider

$y_{app}(x) = \sum_i^n b_i B_i(\alpha,x)+y_0$, with initial conditions, $y_0 = 2$ & $y_0' = 0$. Putting the approximate solution into Eq. (28), we get

$$\frac{d^{\gamma}}{dx^{\gamma}}\left(\sum_i^n b_i B_i(\alpha,x) + y_0\right) - \left(\sum_i^n b_i B_i(\alpha,x) + y_0\right) + \left(\sum_i^n b_i B_i(\alpha,x) + y_0\right)\left(\sum_j^n b_j B_j(\alpha,x) + y_0\right) + \left(\sum_i^n b_i B_i'(\alpha,x) + y_0'\right)\left(\sum_j^n b_j B_j'(\alpha,x) + y_0'\right) - 1 = 0. \qquad (29)$$

Evaluating Eq. (29) by applying the Caputo fractional derivative on the first term, multiplying both sides of the equation by the elements of the fractional B-polys basis set, $B_l(\alpha,x)$, and then integrating both sides of the equation over the interval [0, R]. The expression in Eq. (29) is transformed into,

$$\sum_i^n b_i \Big[\langle D^{\gamma}B_i(\alpha,x)|B_l(\alpha,x)\rangle - \langle B_i(\alpha,x)|B_l(\alpha,x)\rangle + \sum_j^n b_j \langle B_i(\alpha,x)\, B_j(\alpha,x)|B_l(\alpha,x)\rangle + 2\langle y_0\, B_i(\alpha,x)|B_l(\alpha,x)\rangle + 2\langle y_0'\, B_i'(\alpha,x)|B_l(\alpha,x)\rangle + \sum_j^n b_j \langle B_i'(\alpha,x)\, B_j'(\alpha,x)|B_l(\alpha,x)\rangle\Big] = \langle (1 - y_0{}^2 - y_0'^2 + y_0)B_l(\alpha,x)|B_l(\alpha,x)\rangle. \qquad (30)$$

The following is a nonlinear matrix representation of the above equation,

$$B\,[A - C + D + E + F + G] = W, \qquad (31)$$

Where elements of each matrix in Eq. (31) are given as follows:

$$A = \langle D^{\gamma} B_i(\alpha,x) | B_l(\alpha,x) \rangle = \int_0^R D^{\gamma}(B_i(\alpha,x)) B_l(\alpha,x)\, dx,$$

$$C = \langle B_i(\alpha,x) | B_l(\alpha,x) \rangle = \int_0^R B_i(\alpha,x) B_l(\alpha,x)\, dx,$$

$$D = 2\langle y_0\, B_i(\alpha,x) | B_l(\alpha,x) \rangle = 2\int_0^R y_0\, B_i(\alpha,x) B_l(\alpha,x)\, dx,$$

$$E = 2\langle y_0'\, B_i'(\alpha,x) | B_l(\alpha,x) \rangle = 2\int_0^R y_0'\, B_i'(\alpha,x) B_l(\alpha,x)\, dx,$$

$$F = \sum_j^n b_j\, g_{ijl} = b_j \langle B_i(\alpha,x)\, B_j(\alpha,x) | B_l(\alpha,x) \rangle = b_j \int_0^R B_i(\alpha,x) B_j(\alpha,x) B_l(\alpha,x)\, dx,$$

$$G = \sum_j^n b_j\, g_{ijl}' = b_j \langle B_i'(\alpha,x)\, B_j'(\alpha,x) | B_l(\alpha,x) \rangle = b_j \int_0^R B_i'(\alpha,x)\, B_j'(\alpha,x) B_l(\alpha,x)\, dx, \qquad (32)$$

$$W = \langle (1 - {y_0}^2 - {y_0'}^2 + y_0) B_i(\alpha,x) | B_l(\alpha,x) \rangle = \int_0^R (1 - {y_0}^2 - {y_0'}^2 + y_0) B_i(\alpha,x)\, B_l(\alpha,x)\, dx.$$

To obtain an initial guess for the nonlinear terms, we ignore nonlinear matrices F and G and solve the following linear matrix equation,

$$B\,[A - C + D + E] = W. \qquad (33)$$

Using the initial guess from Eq. (33) and substituting it into Eq. (31), we get a new approximation for the unknown coefficients, which are utilized in the linear combination to construct approximate solution. This process is repeated to obtain a converged solution to the fractional differential equation (28). To solve nonlinear fractional differential equation for different fractional values of γ , an iterative scheme has been used. The convergent solutions are plotted for various fractional order of γ.

Figure 7 shows the absolute error between estimated f(x) and exact (sol) solutions is of the order of 10^{-14}. This is the highest accuracy achieved after choosing a set of 8 fractional polynomials. Further accuracy is attainable by increasing the set of fractional polys, but it requires a larger number of iterations to obtain convergent solutions. In the references [35,39], the absolute error is of the order 10^{-3}. As a result, our technique performs better in terms of computational accuracy. Figure 8 depicts graphs of various fractional orders γ = 2,1.5,1.7,1.9 of nonlinear differential equations. The plots of various approximate solutions to various nonlinear fractional-order differential equations are presented to show the smoothness of the curves in the graphs.

Figure 7: For integral order γ=2 in nonlinear differential equation (28), a 1-D graph of approximation f(x), and the precise solution (sol) is displayed on the left side. It shows that the two solutions essentially overlap each other. The plot on the right shows the absolute error between the precise and approximate results. The accuracy of the results is much higher as compared to other results in the literature.

Figure 8: This graph displays the plots of several approximate solutions for various values of the fractional order γ=2,1.9,1.7,1.5 used in Equation (28). All the results converged after 10 number of iterations.

Tables 8, 9, and 10 show that our method predicts excellent solutions when compared to other existing approximate solutions for various fractional order values [35,39]. The accuracy of our method depends on the number of fractional B-polys and iterations used. We receive convergent values after a few iterations and only converged solutions are reported. Comparisons between the available results are given these tables. In some case present results are superior in accuracy as well as less computational time is involved. The error analysis is also carried out using N = 6, 7 and 8 number of fractional B-polys basis set. The error in the solution sets decreases as we increase number of fractional B-polys for solving fractional order differential equations. The results of this observation are shown the Tables presented in this paper.

Table 8: For an integral value of $\gamma = 2$, the comparison of the fractional B-poly method to other methods [35,39] is presented, where N denotes the number of fractional B-polys in the basis set.

$\gamma = 2$							
	Our results						
x	N = 6	N = 7	N = 8	y_{haar}[35]	y_{vm}[39]	y_{hpm}[39]	y_{exact}[39]
0.0	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000
0.1	1.995004	1.995004	1.995004	1.995004	1.994996	1.995013	1.995004
0.2	1.980067	1.980067	1.980067	1.980067	1.979933	1.980200	1.980067
0.3	1.955336	1.955336	1.955336	1.955336	1.954661	1.956013	1.955336
0.4	1.921061	1.921061	1.921061	1.921061	1.918928	1.923200	1.921061
0.5	1.877583	1.877583	1.877583	1.877583	1.872374	1.882813	1.877583
0.6	1.825336	1.825336	1.825336	1.825336	1.814535	1.836200	1.825336
0.7	1.764842	1.764842	1.764842	1.764842	1.744831	1.785013	1.764842
0.8	1.696707	1.696707	1.696707	1.696707	1.662565	1.731200	1.696707
0.9	1.621610	1.621610	1.621610	1.621610	1.566914	1.677013	1.621610
1.0	1.540302	1.540302	1.540302	1.540302	1.456919	1.6250000	1.540302

Table 9: For fractional order of $\gamma = 1.3$ and 1.5, the comparison of the fractional B-poly method to other methods [35] is shown, where N denotes the number of B-polys in the basis sets.

$\gamma = 1.3$					$\gamma = 1.5$			
	Our Result				Our Result			
x	N = 6	N = 7	N = 8	y_{haar}[35]	N = 6	N = 7	N = 8	y_{haar}[35]
0.0	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000	2.00000
0.1	1.92197	1.96543	1.94497	1.96361	1.97465	1.97496	1.97511	1.97607
0.2	1.84223	1.87730	1.85567	1.90615	1.92798	1.92855	1.92880	1.93585
0.3	1.78373	1.74742	1.73137	1.84812	1.86749	1.86815	1.86843	1.88705
0.4	1.74370	1.60752	1.59675	1.79521	1.79691	1.79762	1.79794	1.83225
0.5	1.71384	1.49123	1.48813	1.75255	1.71985	1.72063	1.72099	1.77573
0.6	1.68703	1.41712	1.42255	1.70324	1.64033	1.64116	1.64153	1.72092
0.7	1.66028	1.38228	1.39164	1.65990	1.56273	1.56353	1.56387	1.66355
0.8	1.63486	1.37065	1.38017	1.61351	1.49145	1.49217	1.49248	1.61749
0.9	1.61419	1.37032	1.38129	1.57941	1.43030	1.43092	1.43119	1.56404
1.0	1.59976	1.37954	1.38997	1.55214	1.38178	1.38228	1.38250	1.51286

Table 10: For fractional order of γ = 1.7 and 1.9, the comparison of the fractional B-poly method to other methods [35] is given, where N denotes the number of B-polys in the basis sets.

	γ = 1.7				γ = 1.9			
	Our Result				Our Result			
x	N = 6	N = 7	N = 8	y_{haar}[35]	N = 6	N = 7	N = 8	y_{haar}[35]
0.0	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000
0.1	1.986884	1.986925	1.986948	1.987162	1.993098	1.993103	1.993107	1.988393
0.2	1.957458	1.957558	1.957610	1.958162	1.974312	1.974326	1.974338	1.969996
0.3	1.915545	1.915685	1.915754	1.917257	1.944746	1.944769	1.944788	1.948187
0.4	1.863133	1.863297	1.863378	1.873808	1.905153	1.905184	1.905208	1.907802
0.5	1.801916	1.802104	1.80220	1.820153	1.856259	1.856260	1.856326	1.862716
0.6	1.733598	1.733812	1.73392	1.768749	1.798833	1.798876	1.798912	1.811744
0.7	1.659979	1.660211	1.660326	1.705186	1.733716	1.733766	1.733806	1.741617
0.8	1.582999	1.583239	1.583359	1.640477	1.661831	1.661886	1.661930	1.680558
0.9	1.504768	1.505014	1.505137	1.570663	1.584186	1.584246	1.584294	1.597211
1.0	1.427572	1.427817	1.427939	1.506804	1.501885	1.501949	1.502000	1.526272

Conclusion

The one-dimensional (1-D) fractional-order nonlinear differential equations are solved in this study utilizing the fractional Bhatti polynomial bases set. It is shown that this method of fractional B-polys works well to solve such type of equations and the accuracy of the results can be adjusted by increasing the size of the basis set. The method is applied to four 1-D nonlinear fractional differential equations (NFDEs) with the imposition of initial and boundary conditions. The method can predict highly accurate solutions. In addition to its efficacy, the predicted results are compared to other methods [35,37,39]. The method's versatility is shown to solve various types of differential equations with various values of the fractional order. The solution of the fractional-order differential equation is expanded in terms of linear combination of coefficients which are determined using the Galerkin scheme [40]. In all the 4 examples considered, the linear matrix equation is attained by setting the nonlinear terms of the equation equal to zero. We solve the linear part of the differential equation to obtain initial guess for the unknown coefficients of column matrix B. For example, see Eq. (33), the nonlinear matrix equation is inverted to obtain a new guess in the iteration process, Eq. (31). The iteration procedure is continued until convergent values of the expansion coefficients are obtained. Final expansion coefficients are used to approximate the solution (Eq. 3) of the 1-D fractional-order nonlinear differential equations.

Figures 1 through 8 show the graphs of the convergent solutions of the NFDEs. The error analysis is also carried out for both integral and fractional order differential equations and is presented in the Tables with basis sets. It is clear as the number of iterations is increased the results converge quickly after about 10 iterations and by increasing the fractional B-poly basis set, the accuracy increases rapidly. We compare our solutions to the exact solutions for nonfractional cases and found that the agreement is excellent between fractional order cases, too. Our absolute errors between approximate and exact solutions range between and in all the examples. The results for fractional-order cases are significantly better as compared to other references [35-37,39]. We revealed that our state-of-the-art method could solve variety of examples with greater precision as reported in Tables 2-10 and Figures 1-8. In this article, we provide the graphs for different fractional orders in Figures 2, 4, 6, and 8. All calculations including integrations, fractional differentiations, iterations of results, and matrix inversions are carried out using Wolfram Mathematica symbolic program version-13 [34].

Our method revealed a strong potential for solving nonlinear fractional-order differential equations with a higher degree of precision, easy to use and easy to implement. The computing time for solving examples of integral-order differential equations is less than a minute. For fractional orders examples, the computing time is about 10-30 minutes for accomplishing converged results.

Author Contributions

Conceptualization, M.I.B.; methodology, M.I.B.; software, M.I.B.; validation, M.I.B., N.D., and M.H.R.; formal analysis, M.I.B., and M.H.R.; investigation, M.I.B., and M.H.R.; resources, M.I.B.; data curation, M.I.B., and M.H.R.; writing—original draft preparation, M.I.B.; writing—review and editing, M.I.B., N.D., and M.H.R.; supervision, M.I.B.; project administration, M.I.B and N.D. All authors have read and agreed to the published version of the manuscript.

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Data Availability Statement

Within the article of this study, the data that supports the findings are available.

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Conflicts of Interest

The authors declare no conflict of interest.

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