Acta Scientific Applied Physics

Volume 2 Issue 11 November 2022

Review Article

Unsuspected Statistical Traits of Interacting Fermions

A Plastino[1]* and D Monteoliva[1,2]

[1]*Instituto de F´ısica La Plata–CCT-CONICET Universidad Nacional de La Plata, Argentina*

[2]*Com. de Investigaciones Cient´ıficas Provincia de Buenos Aires, Argentina*

***Corresponding Author:** A Plastino, Instituto de F´ısica La Plata–CCT-CONICET Universidad Nacional de La Plata, Argentina.

Received: October 20, 2022

Published: October 31, 2022

Abstract

We consider finite numbers N of interacting fermions and address the question of their order- disorder properties.. Two well known fermion-fermion interactions are employed: spin-flip and pairing. The second is responsible for superconductivity. In considering their order-disorder properties we look specially at their dependence on N. One finds that, for special N values, unsuspected features emerge.

Keywords: Interacting Fermions; Nuclear Interactions; Spin-Flip; Superconductivity; Order Properties

Introduction

This effort investigates special fermion features focused on the order–disorder opposition, that is specified via a special statistical quantity called the disequilibrium D. We consider SU2xSU2 fermion models and are interested, as stated, in their thermodynamic traits. More specifically, in addition to D, we concentrate attention upon the notion of the energy cost of changing the prevailing order-situation as the interactions' coupling constants vary. We work in a a thermal quantum statistical scenario (Gibbs' canonical ensemble).

The entropy concept is intimately connected to the notion of disorder. The problem is how to quantify order. We appeal to a special order quantifier called disequilibrium D (consult [1] and references therein). D is an Euclidean distance in probability space: that between the current probability distribution and the uniform one. Of course, the latter is considered to be the maximally "disordered" one. Consequently, the largest D value ought to correspond to the biggest possible degree of order.

We use two simple and exactly solvable fermion models that are known to permit interesting insight into the intricacies of the quantum many-fermion problem, avoiding dealing with huge Hamiltonian matrices [2-4]. It was extraordinarily helpful in fermions theoretical research to employ the exactly solvable Lipkin Model (LM) [2-7]. This shed light on manifold traits of the many-body problem [8]. The two-energy-level LM (containing N fermions) is based on an SU2 algebra generated by operators denominated quasi-spin ones. The models of this paper are Lipkin model variants that possess exact analytical solutions (note that the LM model requires numerical diagonalization).

The angular momentum Casimir operator (CO) J^2 permeates both the LM [5] and its present variants. N fermions are allocated to two N degenerated singe-particle levels. Our CO possesses different multiplets. There exists an unperturbed ground state multiplet of great importance [5]. Let us insist in that each of our two energy levels is degenerate and capable by itself to hold all our $2\Omega = N$ fermions. Thus, each level has N loci or sites, specified by a quasi spin index p. A site can be either occupied or empty. We denominate sister sites those loci with the same p value (p runs from unity to 2Ω, of course).

Cambiaggio and Plastino (CP) [1,9] proposed years ago an SU2XSU2 Lipkin extension that permits treating the excited Lipkin multiplets, which was not possible before. This extension allowed one to formulate in quasi-spin language a kind of BCS formalism that mimics nuclear superconductivity [10-12], yielding exact analytic results. In such CP-model, the BCS solution is identical to the exact solution. The CP formalism entails a variable particle number.

We will proceed on this basis, as specified below.

Present program of activities

- To determine the form of a statistical indicator of order, the disequilibrium D.
- To determine how D depends on the number of fermions N.
- To determine how much free energy F one needs to modify D. The D changes we are interested in ensue as a result of variations in the coupling constants of the fermion-fermion interactions.
- The accompanying free energy costs of the D change are specified via a new statistical quantifier ν (order variation cost).
- We deal with two different interaction terms in the Hamiltonian H. They possess coupling constants V and G, respectively. Thus, we have two different kinds of ν, called $\boldsymbol{\nu}_V$ and $\boldsymbol{\nu}_G$. They reflect the cost of changing either of the two coupling constants.
- We study the behavior of D, $\boldsymbol{\nu}_V$, and $\boldsymbol{\nu}_G$ as a function of the number of fermions N and obtain interesting results.

Present fermions-system

Our SU2xSU2 Hamiltonian used in this effort has three components. 1) An unperturbed Hamiltonian H_0 proportional to the z-compoenet of the angular momentum J_z plus two distinct interactions: 2) a pairing one H_{CP} (with a dimensionless coupling constant G) [1] and 3) a spin–flip one [3] (with a dimensionless coupling constant V), so that the Hamiltonian reads

$$H = H_0 + H_M + H_{CP}. \qquad (1)$$

Our thermal elaborations are performed in Gibbs' canonical ensemble at the inverse temperature $\beta = \boldsymbol{\alpha} T^{-1}$. One detects two T = 0 attractors (one for H_{CP} and the other for H_M) [13,14]. For the M attractor, each sister pair of sites is singly occupied. In the CP attractor, each sister pair is either doubly occupied or empty. People regard such a situation as an "ordered" one and, on the other hand, consider "disordered" those cases in which the two energy levels have different occupation numbers.

The LMC scheme

For our purposes, as anticipated in the Introduction, the protagonist is the disequilibrium D [15-18]. We know that D is associated with a uniform PD (P_{unif}). D measures the Euclidean distance from the probability distribution (PD) of interest to P_{unif} [15,16]. A hierarchy is established by the numerical values that D assumes, which reflects the existence of "privileged" states among the accessible ones to our system [15,16]. If we multiply D by the entropy S one obtains a new quantifier called the statistical complexity C [15].

$$C = DS. \qquad (2)$$

It is popular today to assert that C grasps the system's structural traits in the same manner that entropy does with randomness [15]. For a case in which one has M accessible single particle states D is expressed as [15,16].

$$D = \sum_{i=1}^{M} \left(p_i - \frac{1}{M} \right)^2, \qquad (3)$$

Where {p1, p2, . . . , pM } are the normalized probabilities ($\sum_{i=1}^{M} p_i = 1$) [15]. D gets its maximum possible value for a perfectly ordered state and, of course, vanishes for P_{unif}. By the way, remember that

$$J_z = (1/2) \sum_{p,\mu} \mu C^{+}_{p,\mu} C_{p,\mu}, \qquad (4)$$

The trio S, D, and C constitutes the so-called LMC-scheme. It has received great attention, with a lot of applications [15-18].

Quasi spin formalism

As we have mentioned before, we will consider a spin–flip interaction [3,14] and a pairing one [1]. The second interaction mimics nuclear superconductivity [10]. Our dealings have an SU2XSU2 substratum that presupposes N fermions apportioned to 2 Ω-fold degenerate single-particle (sp) levels (N = 2 Ω). Our two energy levels are separated by a an energy gap ϵ = 1 (arbitrary units). The system's sp states are characterized by two quantum numbers: p, μ, with p = 1, . . . , 2Ω, and μ = ±1. p denotes both a quasi-spin quantum number and also a "site" or locus [5]. The

scheme necessitates SU2 quasi-spin operators J^i [5] cast in terms of creation and annihilation operators C^+, C in the fashion.

$$J_+ = \sum_p C^+_{p,+} C_{p,-}, \quad (5)$$

$$J_- = \sum_p C^+_{p,-} C_{p,+}. \quad (6)$$

$$Q_0 = (1/2) \sum_{p,\mu} C^+_{p,\mu} C_{p,\mu} - \Omega, \quad (7)$$

We need also angular momentum-like pairing SU2 operators [9]:

$$Q_+ = \sum_p C^+_{p,+} C^+_{p,-}, \quad (8)$$

$$Q_- = \sum_p C_{p,-} C_{p,+}, \quad (9)$$

$$J + Q = \Omega. \quad (10)$$

Where Q+ creates and Q− destroys a pair of fermions, that together yield a null contribution to the J_z value. They are said to be paired to $J_z = 0$. Any J-operator commutes with all Q operators, and vice versa (SU2 × SU2). The pertinent Hilbert orthonormal basis is of the form of $|J^2, J_z, Q^2, Q_0>$. A fundamental relation is [9].

J + Q = Ω.

We set 2J = N [5]. The unperturbed ground state possesses J = Ω, J_z = -Ω, and Q = Q_0 = 0 [9]. The spin flip Hamiltonian is

$$H_{PM} = H_0 + H_M = H_0 - V\left[J^2 - J_z^2 - N/2\right], \quad (11)$$

(Coupling constant V). If $J_z|J, M\rangle = M\,|J, M\rangle$, then $J^2|J, J_z\rangle = J(J + 1)|J, J_z\rangle$:

$$H_0 = J_z, \quad (12)$$

Where the eigenvalues of J_z are named M. The HP M -eigenvalues read [14]

$$E(J, J_z) = M - V\left[J(J+1) - M^2 - N/2\right]. \quad (13)$$

The energy of the unperturbed (V = 0) gs (ugs) (Q = Q_0 = 0) is

E0 = −Ω. (14)

The all important feature of HP M is that, as V increases, the system undergoes Ω level crossings. Indeed, the ground state (at T = 0) stops being characterized by J_z = −J at V = V_{crit} = $(N - 1)^{-1}$. As the coupling constant grows it comes to be linked with increasingly larger J_z values until we arrive at M = 0 for V = 1 [14]. The state |J, M = 0⟩ is an "attractor" for the system as V is grows (at T = 0).

Let us pass now to the pairing interaction G/2 Q+Q− that originates superconductivity,

$$H_{CP} = H_0 - \frac{G}{2} Q_+ Q_-, \quad (15)$$

Adding to J_z the pairing contribution E_p

$$E_P = -(G/2) Q(Q+1). \quad (16)$$

$H_0 + H_{CP}$ exhibits also a phase transition at G = G_{crit} = $(\Omega - 1)^{-1}$ [9]. There the system becomes a superconductor [9]. H_{CP}, for large enough G, exhibits a second T = 0 "attractor" state. This superconducting state is associated with M = 0 with half of the p sites being doubly occupied.

Statistical mechanics treatment of our two interactions

For this we require the math results provided by [19]. Note that, while for the ground state at T = 0 we just use the J + Q = Ω multiplet or band, when T = 0 states belonging to other bands play a role. The degeneracy Y (J, Q) is, if β = 1/T,

$$Y(J, Q) = \frac{(2\Omega + 2)!(2\Omega)!(2J + 1)(2\Omega + 1)}{(\Omega + J + Q + 2)!(\Omega + J - Q + 1)!} \quad (17)$$

We start with a partial partition function ZM that runs only over M reads [19]

$$Z_M(\beta) = \sum_{M=-J}^{M=J} \exp\{-\beta[[M - V(J(J+1) - M^2 - J)] - (G/2) Q(Q+1)]\}. \quad (18)$$

Instead, the total partition function Z is

$$Z(\beta) = \sum_{J,Q} Y(J, Q) Z_M(\beta). \quad (19)$$

J and Q run over values allowed by the SU2 × SU2 symmetry [19]. One has

$$0 \le J \le \Omega, \quad (20)$$

$$0 \le Q \le \Omega, \quad (21)$$

$$0 < J + Q \le \Omega. \quad (22)$$

We also need a slight change in (22). Now [19],

0 ≤ J + Q = s ≤ Ω.

We sum over J + Q = R and then over J, with Q fixed at Q = R − J. Then we have [19]

$$0 \le R \le \Omega. \quad (23)$$

One has R = 0, 1, 2, 3, . . . , Ω while J = 0, 1, 2, , R and

$$Z = \sum_{J,Q} Y(J,Q) Z_M = \sum_{R=0}^{\Omega} \sum_{J=0}^{R} Y(J, R-J) Z_M. \quad (24)$$

Remember that N = 2Ω [5]. Finally, one has

$$D = \sum_{J,Q} \sum_{M=-J}^{M=J} [P(M,Q) - P_{unif}]^2, \quad (25)$$

And

$$S = -\sum_{J,Q} \sum_{M=-J}^{M=J} P(M,Q) \ln P(M,Q). \quad (26)$$

Order energy cost (OEC) ν

The OEC quantifies the energy needed to change D and we want ν(N). We have here two dimensionless coupling constants VX1 and GX2. A change from X to X + dX originates a variation in D after performing some amount of work W on the system. In the wake of [20], we define

$$\nu(X; dX) = \frac{dD}{dW}, \quad (27)$$

Where dW is the work performed on the system as a result of the dX change. This work is given in terms of the free energy F = E - TS. Now ν(X; dX) is the diminution (increase) in D emerging from each unit of work performed on the system. If dF is negative, it is the system itself that does the work. We always think of quasi-static processes.

Our X(N) and also D(N) are determined by alternatively fixing

V (and setting G = 0)

G (and setting V = 0),

And we have

$$\nu(X; dX)(N) = \frac{\partial D}{\partial F}(N) = [\frac{\partial D}{\partial X} / \frac{\partial F}{\partial X}](N). \quad (28)$$

In more detail,

$$\nu_G(N) = \frac{\partial D}{\partial F}(N) = [\frac{\partial D}{\partial G} / \frac{\partial F}{\partial G}](N), \quad (29)$$

$$\nu_V(N) = \frac{\partial D}{\partial F}(N) = [\frac{\partial D}{\partial V} / \frac{\partial F}{\partial V}](N). \quad (30)$$

To repeat: the modifications in F can be connected to the work performed on the system dF = dW [20]. If $\boldsymbol{\nu}_V$ (alternatively $\boldsymbol{\nu}_G$) diminishes, it requires energy for this to take place.

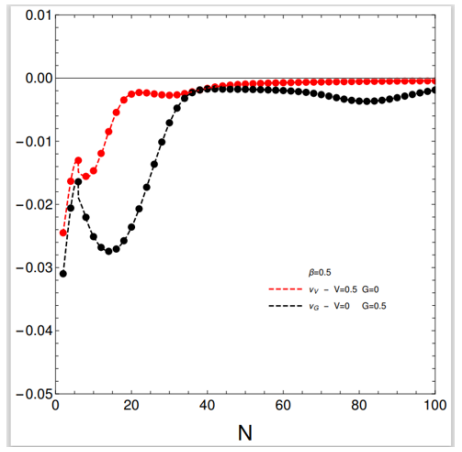

Figure 1: Plot of D versus N for β = 0.5. The red line is the spin flip disequilibrium D, while the blue one corresponds to the pairing interaction (originating superconductivity) D. The main result is that the system becomes the more ordered the larger N (disordered for few fermions). The pairing interaction creates more order than the spin flip one. Note that artifacts are observed only for N smaller than 50. We can mention than heavy nuclei tend to be superconductors. Our results seem to agree with this empiric fact.

Results

Remind that S, F, D, ν, etc., all depend on both V and G. At finite temperature, a change dX in X emerging from, say dG in G, will be also linked to the actual V value: dX = dX(G, V).

There exists a statistical correlation V − G in

$$dX = dX(G,V), \quad (31)$$

An artifact of finite temperatures. Such an interconnection will influence the costs we are interested in here. Our results are displayed in figure 1 (D versus N) and 2 (ν versus N).

We see that order grows as N increases for the two interactions. This is an unsuspected property of interaction fermions. The interaction acts in a fashion that "orders" the fermions-system as N grows. If N is large enough, this order "saturates" and stops increasing. It takes free energy expenditure to change the coupling constants. But this expenditure diminishes as N grows and tends to vanish. Things happen as if the augmenting of N would tend by

itself to tightly bind the fermions, an unsuspected trait. We point out that our two present fermion. Fermion interactions do occur in Nature.

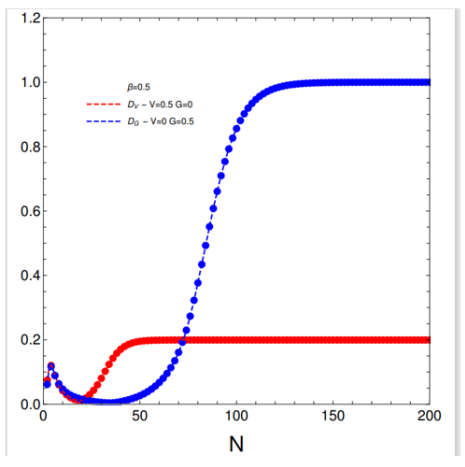

Figure 2: Plot of the energy costs ν versus N for β = 0.5. Red line spin flip one and black line pairing interaction. Both are negative. The system must perform work to change D. This work tends to vanish as N grows. Note that artifacts are observed only for N smaller than 50. It seems natural that once superconductivity is achieved, it takes no work to change the degree of order by increasing N.

Conclusions

In the present work we studied how high the degree of order of a system of fermions becomes when the particles interact either via a spin flip or via a pairing interaction at a finite temperature. We were particularly interested in the behavior-changes when the number of fermions N changes. Our analysis was carried out for exactly solvable models endowed with a Lie structure.

First, we constructed a statistical indicator of order called the disequilibrium D.

We ascertained the behavior of D versus N. We saw that, for both interactions, order grows as the number of fermions augments, an unexpected result.

Also, one ascertained how much free energy F is required to modify D by varying the coupling constants of the interactions. This results in two quantifiers $\boldsymbol{\nu}_G$ and $\boldsymbol{\nu}_V$. We study their behavior as a function of N.

Both $\boldsymbol{\nu}_s$ are negative. They tend to vanish as N grows, another unsuspected result. Once the number of fermions is large enough, it takes no free energy cost to increase N.

Bibliography

1. Pennini F and Plastino A. "Complexity and disequilibrium as telltales of superconductivity". *Physica A* 506 (2018): 828-834.

2. Cervia MJ., *et al*. "Lipkin model on a quantum computer". arXiv arXiv:2011.04097 (2021).

3. Balian R., *et al*. "Temperature Dependence of Even-Odd Effects in Small Superconducting Systems". arXiv arXiv:cond-mat/9802006 (1998).

4. Sedrakian A and Clark JW. "Superfluidity in nuclear systems and neutron stars". *The European Physical Journal A* 55 (2019): 167.

5. Lipkin HJ., *et al*. "Validity of many-body approximation methods for a solvable model: (I). Exact solutions and perturbation theory". *Nuclear Physics* 62 (1965): 188-198.

6. Lerma HS and Dukelsky J. "The Lipkin-Meshkov-Glick model from the perspective of the SU (1,1) Richardson-Gaudin models". *Journal of Physics Conference Series* 492 (2014): 12013.

7. Providencia C., *et al*. "The Lipkin Model in Many- Fermion System as an Example of the su (1,1) times su (1,1)-Algebraic Model". *Progress of Theoretical and Experimental Physics* 116 (2006): 87-105.

8. Nolting W. "Fundamentals of Many-Body Physics". Springer: Berlin/Heidelberg, Germany, (2009).

9. Cambiaggio MC and Plastino A. "Quasi spin pairing and the structure of the Lipkin Model". *Zeitschrift für Physik A Atoms and Nuclei* 288 (1978): 153-159.

10. Ring P and Schuck P. "The Nuclear Many-Body Problem". Springer: Berlin/Heidelberg, Germany (1980).

11. De Llano M., *et al*. "Multiple phases in a new statistical boson fermion model of superconductivity". *Physica A: Statistical Mechanics and its Applications* 317 (2003): 546-564.

12. Uys H., *et al*. "Generalized statistics and high-Tc superconductivity". *Physics Letters A* 289 (2001): 264-272.

13. Plastino A Ferri GL and Plastino AR. "Spectral explanation for statistical odd-even staggering in few fermions systems". *Quantum Report*s 3 (2021): 166-172.

14. Plastino A and Moszkowski SM. "Simplified model for illustrating Hartree-Fock in a Lipkin- model problem". *Il Nuovo Cimento A (1965-1970)* 47 (1978): 470-474.

15. L´opez-Ruiz R., *et al*. "A statistical measure of complexity". *Physics Letters A* 209 (1995): 321-326.

16. L´opez-Ruiz R. "Complexity in some physical systems". *International Journal of Bifurcation and Chaos* 11 (2001): 2669-2673.

17. Rudnicki L., *et al*. "Monotone measures of statis- tical complexity". *Physics Letters A* 380 (2016): 377-380.

18. L´opez-Ruiz R., *et al*. "A Statistical Measure of Complexity in Concepts and Recent Advances in Generalized Information Measures and Statistics". Kowalski, A., Rossig- noli, R., Curado, E.M.C., Eds.; Bentham Science Books: New York, NY, USA, (2013): 147-168.

19. Rossignoli R and Plastino A. "Thermal effects and the interplay between pairing and shape deformations". *Physical Review C* 32 (1985): 1040-1048.

20. Nigmatullin R and Prokopenko M. "Thermodynamic efficiency of interactions in self-organizing systems". *Entropy* 23 (2021): 757.