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Review Article

Application of Thermal Statistical Tools to Newton's Gravity

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Abstract

We develop a statistical physics' picture of classical Newtonian gravitation and uncover rather notable en- tropic features. In particular, applying the classical canonical ensemble treatment of Newtonian gravitation (NG) generates unsuspected statistical constraints on important physical quantities entering the thermo- dynamics of gravitation. Some of these quantities are, for instance, the masses involved. We work in Gibbs' canonical ensemble. One must appeal for this to a generalization of the Dimensional Regulation approach of Bollini and Giambiagi. For a full explanation of this theory see Dimensional Regularization and Non-Renormalizable Quantum Field Theories. Cambridge Scholars Publishing (2021). ISSN: 1-5275–6395-2.

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Introduction

In this work we develop some aspects of the Newtonian theory of gravitation.

For this purpose, we resort to the physical and mathematical theories described in [1], which have already been used in several articles (see, for example [2-5]).

The entropy notion is associated to diverse ideas, in particular, ignorance or disorder. We will concentrate here on the first one. Our focus is classical Newton gravity's workings when statistically investigated by appeal to Gibbs' canonical ensemble techniques.

Entropy and ignorance

Entropy is lack of information, i.e., ignorance [6]. How does this ignorance manifests itself physically? In variegated ways, of course. Just to fix ideas, we concentrate our attention now in an emblematic example: the Harmonic Oscillator (HO). The basic physical quantity one has to know in dealing with an HO is its frequency ω.

Let us begin with the quantum statistics of the HO at the temperature T [7-9]. Let k_B stand for Boltzmann's constant, β for the inverse temperature T, and $e_\pm = \exp[\pm\beta\hbar\omega]$. Then we have for the entropy the expression.

$$S_{qHO} = -k_B \ln(1-e_-) + (\hbar\omega/T)[\frac{e_-}{1-e_-}]. \quad (1)$$

It diverges if $\omega = 0$. Our ignorance is infinite. Consider now the classical HO [8]. We have that also diverges for $\omega = 0$. We see then that if the critical quantity for the theory vanishes, the ignorance augments without bounds. This also happens in an interesting paper on coupled harmonic oscillators, recently published, if suitable constants of that paper called $C_1 = C_2 = C_3$ vanish, as we are left only with kinetic energy [10].

$$S_{cHO} = -k_B \ln\left(\frac{\hbar\omega}{k_B T}\right) + constant, \quad (2)$$

Problem and goal

We will be concerned in this paper with classical Newtonian gravity. No relativistic consideration is needed. Just classical statistical mechanics is enough. The critical piece of knowledge is the gravitational constant G, or more properly, as we will soon realize, the quantity $x = Gm_1m_2/k_BT$. The m's are the interacting masses, T the temperature, and k_B Boltzmann's constant.

We ask now what happens with our gravity's entropic ignorance when our critical piece of knowledge vanishes ($x = 0$)? The answer is much more complicated in this scenario than for the HO, as we discuss in the forthcoming Section. Our goal, in order to answer the question, is to adequately face and successfully deal with this complicated classical layout.

Accompanying mathematical troubles

As it is well known, appealing to conventional integration tools the classic gravitational thermodynamic functions turn out to be not finite [[11] (and references therein)]. This difficulty can be circumvented by using special, rather advanced mathematical techniques [11]. This involves using a combination of

- A generalization of the dimensional regularization [11] and
- The analytical extension of an associated ensuing integral made by Gradshtein and Rizik in their celebrated Table [12].

We begin with the handling of these items next.

Mathematical details

As stated above, here we must appeal to a rather not conventional mathematical apparatus for adequately analyzing the ensuing classical scenario.

Our classical scenario

We envision a classical canonical ensemble whose constituents are two gravitationally interacting masses m_1 and m_2 at the inverse temperature $\beta = 1/k_BT$. It is usual to pass to a center-of-mass M and relative one m, separated by a distance r. The gravitation constant is G.

The classical concomitant partition function Z (in any number ν of spatial dimensions) diverges [11], and one has (x-p) as the phase-space coordinates in the special case of spatial dimension ν = 3.

$$Z = \int_M e^{-\beta\left(\frac{p^2}{2m} - \frac{GmM}{r}\right)} d^3x d^3p. \quad (3)$$

Most people believe that Z diverges. However, such belief does not take into account the possibility of analytical extensions, that would take care of divergences, e.g., at the origin. A useful result obtained in [11] is the integral that is needed for the forthcoming developments.

$$\int_0^\infty x^{3-1} e^{-\beta x^2} dx = \frac{2^{-3}\beta^{-\frac{3}{2}}\sqrt{\pi}\,\Gamma(3)}{\Gamma\left(\frac{3+1}{2}\right)}, \quad (4)$$

Our classical 3D partition function Z = Z3

In tackling the Z integration process we employ hyper-spherical coordinates. One faces two integrals, each in 3 spatial dimensions. A change of variables is mandatory. We work with

$$x_1 = r\cos\theta_1$$
$$x_2 = r\sin\theta_1\cos\theta_2$$
$$x_3 = r\sin\theta_1\sin\theta_2\cos\theta_3 \quad .$$
$$x_{3-1} = r\sin\theta_1 \ldots\ldots \sin\theta_{3-2}\cos\theta_{3-1}$$
$$x_3 = \sin\theta_1 \ldots\ldots \sin\theta_{3-1}\sin\theta_{3-1}, \quad (5)$$

Where $0 \le \theta_j \le \pi$, $1 \le j \le 3 - 2$, and $0 \le \theta_3 - 1 \le 2\pi$. Integrating over the angular variables ($\Omega_3 = (\theta_1, \theta_2, \theta_{3-1})$) one finds

$$\int_{\Omega_3} d\Omega_3 = \left[\frac{2\pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}\right]. \quad (6)$$

We face just two radial coordinates (one in r− space and the other in p− space) and 4 angles. Thus,

$$z = \left[\frac{2\pi^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}\right]^2 \iint_0^\infty (rp)^2 e^{-\beta\left(\frac{p^2}{2m} - \frac{GmM}{r}\right)} dr\, dp. \quad (7)$$

We appeal now to (4). Accordingly, from

$$z_3 = 4\sqrt{\pi}\cos(\pi 3)\left(\frac{\pi^2\beta G^2 m^3 M^2}{2}\right)^{\frac{3}{2}} \frac{\Gamma(3)\Gamma(-3)}{\left[\Gamma\left(\frac{3}{2}\right)\right]^2 \Gamma\left(\frac{3+1}{2}\right)}. \quad (8)$$

Eq. (8) is telling us that, regretfully, poles have emerged above. The day is saved here by appeal to the so-called dimensional regularization (DR) approach, as explained in reference [11].

Regularized gravitational partition function Z

Focus attention upon (8). The central idea of our procedure is simply explained. If we have a quantity F (ν) that diverges for special values of the dimension ν (here ν = 3), the DR generalized approach consists in performing the Laurent-expansion of F around ν = 3 and select after wards, as the physical result for F, the ν = 3-independent term in the ensuing expansion. The reasons for such a procedure are discussed in detail in [11].

Here, the pertinent Laurent expansion in the variable ν around ν = 3 is [[11] and references therein].

$$Z_\nu = -\frac{2}{3\sqrt{\pi}}\frac{(2\pi^2\beta G^2m^3M^2)^{\frac{3}{2}}}{3(\nu-3)} - \frac{1}{3\sqrt{\pi}}(2\pi^2\beta G^2m^3M^2)^{\frac{3}{2}}\otimes$$
$$\left[\ln\left(2\pi^2\beta G^2m^3M^2\right) - \boldsymbol{C} - \frac{17}{3}\right] + \sum_{s=1}^{\infty} a_s(\nu-3)^s. \qquad (9)$$

Where C is Euler's constant. Z_ν diverges at ν = 3. By definition (this is the essential aspect of DR), the independent (ν 3)-term in the Z_ν-Laurent expansion will give the physical value of Z_ν, as explained in [2]. Accordingly,

$$Z = -\frac{1}{3\sqrt{\pi}}(2\pi^2\beta G^2m^3M^2)^{\frac{3}{2}}\left[\frac{17}{3} - \boldsymbol{C} - \ln\left(8\pi^2\beta G^2m^3M^2\right)\right]. \qquad (10)$$

Let c_i stand for several constants (independent of either β or of GMm that appear above). By inspection then, we can write the partition function Z in the more useful form.

$$Z = c_1(\beta G^2m^3M^2)^{3/2}[c_2 - \ln\left(c_3\beta G^2m^3M^2\right)], \qquad (11)$$

Or, setting

$$x = \beta G^2m^3M^2, \qquad (12)$$

Which will be an important physical variable here on, we have for the partition function

$$Z = c_1x^{3/2}[c_2 - \ln\left(c_3x\right)], \qquad (13)$$

So that the all important quantity given by the logarithm of Z, that is proportional to Helmholtz free energy

F, reads

$$\ln Z = \ln c_1 + \frac{3}{2}\ln x + \ln\left[c_2 - \ln c_3 - \ln x\right]. \qquad (14)$$

First classical statistical task: avoiding imaginary entropic values

Now we have Helmholtz' free energy F

$$F = -k_BT\ln Z = -k_BT[\ln c_1 + \frac{3}{2}\ln x + \ln\left[c_2 - \ln c_3 - \ln x\right]], \qquad (15)$$

And we construct the associated entropy S via

$$S = -dF/dT. \qquad (16)$$

Reintroducing the constants' proper values we finally have

$$S = \ln\left\{\frac{1}{3\sqrt{\pi}}(2\pi^2x)^{\frac{3}{2}}\left[\ln\left(8\pi^2x\right) + 3\boldsymbol{C} - \frac{17}{3}.\right]\right\} +$$
$$\frac{3\left(3C + \ln\left(8\pi^2x\right) - 5\right)}{2\left[\ln\left(8\pi^2x\right) + 3\boldsymbol{C} - \frac{17}{3}.\right]} \qquad (17)$$

We plot in figure 1 S(x) versus x. Note that the vertical axis S is NOT plotted from S = 0 but from S ~ 5. S diverges at x 0.65. This is so because, at this x value, the denominator in the second term of the r. h. s. of (17) vanishes. Also, for x smaller than that very value, the logarithm in the first term there becomes imaginary (imaginary entropy!). Thus, necessarily x ~ 0.65. The typical quantity of statistical gravity, x, cannot be zero, a significant peculiarity. Of course, such condition restricts, from a theoretical viewpoint, the possible values of m, M, G, and T. This is our most basic result here, i.e., thermal gravity restrictions on the possible values of important and typical physical quantities. Figure 2 illustrates the S versus G comportment.

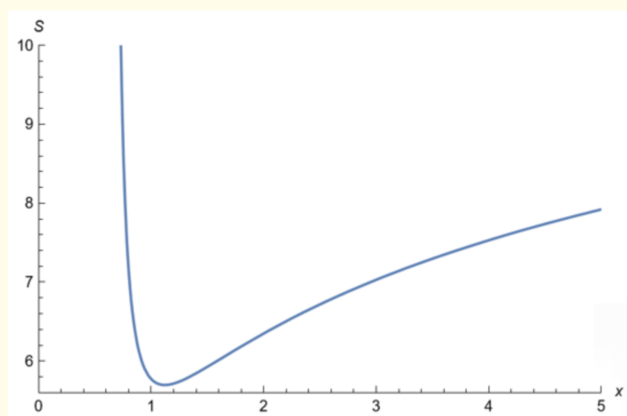

Figure 1: Plot of S versus x. Notice the origin of S is not displayed. The minimum of S is NOT equal zero but is located at S ~ 7.6. S diverges at x ~ 0.65, as explained in the text.

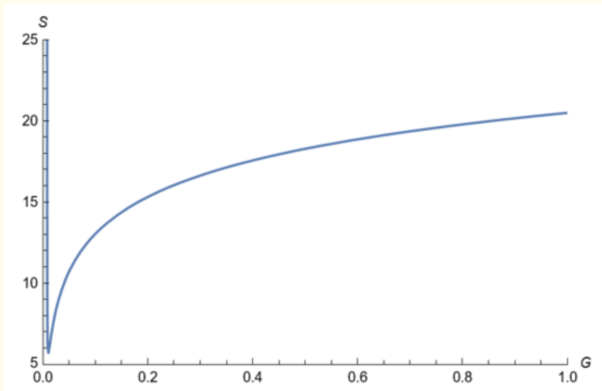

Figure 2: Plot of S versus G for m = 1, M = 100, β = 1. Note the divergence at the extreme left which indicates x = 0.65.

Emerging constrains of our 3D-classical entropy treatment

We started our proceedings looking for an answer to the question: What happens with the entropic ignorance when the critical quantity x of Newtons' gravity vanishes? After obtaining Z and then S we realized that the entropy might be imaginary for certain x values. As stated above, to avoid this the argument of the logarithm in the first term of (17) above must be greater than zero. Otherwise, of course, S might have an imaginary component. Thus, we found the following important inequality (remember that C is Euler's constant, ≃ 0.577).

$$x = \beta G^2 m^3 M^2 \geq \frac{\exp(17/3)}{8\pi^2 e^C} \sim 0.65, \quad (18)$$

That carries very important (and hopefully new) classical gravitational information.

Answer to our initial query

Our initial question was: what happens if x vanishes? The rather surprising response is that x can not vanish according to Eq. (18). The entropy (ignorance) diverges, but not at x = 0 but at x ~ 0.65.

Further statistical constraints

Further, from (18) we obtain the additional restrictions on G and T

$$G \geq \left[\frac{e^{\frac{17}{3}-C}}{8\pi^2 \beta m^3 M^2}\right]^{\frac{1}{2}}, \quad (19)$$

And

$$T \leq \frac{e^{C-\frac{17}{3}} 8\pi^2 m^3 M^2 G^2}{k_B}. \quad (20)$$

We plot next T versus G in figure 3, for M = 100 and m = 1.

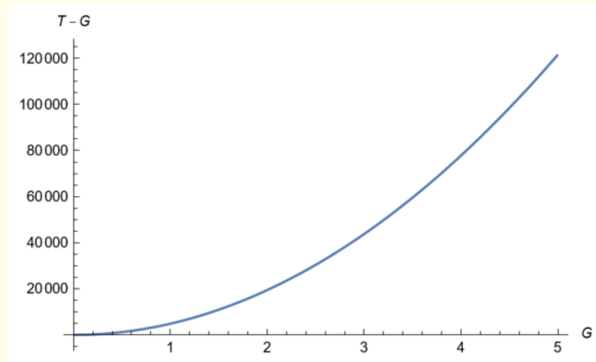

Figure 3: T versus G. We draw the T - G curve at which the entropy diverges according to Eq. (4.12). We set m = 1, M = 100. Note that the origin can not be reached on account of the inequality x <~ 0.65.

Summarizing, we have obtained hopefully new statistical information regarding statistical constraints to Newton's gravity workings:

- Neither G nor the masses can be arbitrarily small.
- T can not be arbitrarily large.
- If the masses grow, the lower bound for G diminishes.
- x is the basic statistical quantity for gravitation. Since the cosmic background radiation's temperature evolves with time, so does x.
- S grows as T augments, as it should.
- S diverges as T → 0. This is a peculiar information theoretic effect of classical Newtonian gravity.

Conclusion

In this work, we have evaluated classical thermodynamic functions corresponding to Newton's three-dimensional gravity. Since the integral that defines the partition function is divergent, we have used a generalization of the dimensional regularization approach of Bollini and Giambiagi. Some seemingly important new statistical information has been gathered.

It was suggested by Dirac in 1937 that G might vary with time [13]. Here we showed that the typical quantity of thermal statistical classical gravity is the one we called x above, that depends upon the masses, G, and T. Since the Big Bang epoch, the cosmic background temperature has diminished, which entails that x has been allowed to grow with time. Thus, at least some gravitation-related quantity seems to depend on time.

Availability

All that might be needed is included in the present manuscript.

Conflict of Interests

None.

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