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On Hermite Polynomials and their Generalizations

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Abstract

The article is written with the objective of carrying out a study on Hermite polynomials and their respective generalizations. Additionally, we will show the applications they have in the field of special functions and their relations with Bernoulli polynomials, Euler polynomials, Genocchi polynomials and Frobenius-Euler polynomials.

Keywords: Bernoulli Polynomials; Hermite Polynomials; Genocchi Polynomials and Euler Polynomials

On hermite polynomials

Throughout this paper, we use the standard notions: N:= {1, 2, . . . ;} N_0: = {0, 1, 2, ...}. Respectively, Z, R and C, are called the set of integer, real and complex numbers.

The multi-variable Hermite polynomials were introduced by Hermite himself some time ago. These polynomials are been used in many branches of mathematics, such as in the analysis and theory of special functions as well as formulation of the quantum mechanics of space. These polynomials gave new opportunities to explore and extend the theory of special function. The introduction of new families of special polynomials such as, Hermite-Euler polynomials, Hermite-Bernoulli polynomials, Hermite-Frobenius-Euler polynomials and Hermite-Genocchi polynomials have been possible thanks to the multi-variable forms of these special polynomials of mathematical physics that as consequence help obtained applicable identities to these new generations of polynomials.

The 2-variable Kampé de Fériet generalization of the Hermite polynomials are given by (see, [2]):

$$H_n(x, y) = n! \sum_{n=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!}. \qquad \text{-----------(1)}$$

It is to be noted that

$$H_n(2x, -1) = H_n(x),$$

Where $H_n(x)$ are the ordinary Hermite polynomials [1].

The already stated polynomials, please with exactitude the following generating equation:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y)\frac{t^n}{n!}. \qquad \text{---------(2)}$$

In [4], they define the three-variable Hermite polynomials (3V HP) $H_n(x, y, z)$ as:

$$e^{xt+yt^2+zt^3} = \sum_{n=0}^{\infty} H_n(x, y)\frac{t^n}{n!}, \qquad \text{-------------(3)}$$

Which for z = 0 reduce to the two-variable Hermite-Kampé de Fériet polynomials $H_n(x, y)$ and for z = 0, x = 2x and y = −1 become the classical Hermite polynomials $H_n(x)$.

The classical Bernoulli polynomials Bn(x), are defined as (see, [6] p. 61):

$$\frac{t}{e^t-1}\, e^{tx} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad |t| < 2\pi, \qquad \text{-------(4)}$$

For the classical Bernoulli numbers Bn, we readily find from (4) that $B_n := B_n(0) = B_n(0)$, $(n \in N_0)$.

The classical Euler polynomials $E_n(x)$, are usually defined as follows (see, [8]):

$$\frac{2}{e^t+1}\, e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad |t| < \pi, \qquad \text{------------(5)}$$

For the classical Euler numbers E_n, we readily find from (5) that $E_n := E_n(0)$, $(n \in N_0)$.

The classical Genocchi polynomials $G_n(x)$, are defined as (see, [8]):

$$\frac{2z}{e^t+1}\, e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \quad |t| < \pi, \qquad \text{------- (6)}$$

For the classical Genocchi numbers G_n, we readily find from (6) that $G_n := G_n(0)$, $(n \in N_0)$.

For $u \in C$, $u = 1$, the generating equation for the Frobenius-Euler polynomials is given by (see, [5]):

$$\frac{1-u}{e^t-u}\, e^{xt} = \sum_{n=0}^{\infty} H_n(x;u)\frac{t^n}{n!}, \qquad \text{---------(7)}$$

For $u = -1$, the reduce to the classical Euler polynomials $E_n(x)$.

Taking into account the generating functions given in (2), (3), (4), (5), (6) and (7), many authors have undertaken the task of studying new families of Hermite-type polynomials such as the following, the Hermite-Bernoulli polynomials, the Hermite-Euler polynomials, the Hermite-Genocchi polynomials and Hermite-Frobenius-Euler polynomials, given by the following generating functions (see, [3,7,9]).

$$\frac{t}{e^t-1}\, e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x)\frac{t^n}{n!}, \qquad \text{------------(8)}$$

$$\frac{2}{e^t+1}\, e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE_n(x)\frac{t^n}{n!}, \qquad \text{--------------------(9)}$$

$$\frac{2t}{e^t+1}\, e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HG_n(x)\frac{t^n}{n!}, \qquad \text{--------------(10)}$$

$$\frac{1-u}{e^t-u1}\, e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HH_n(x;u)\frac{t^n}{n!}. \qquad \text{-------------(11)}$$

In these works, a multi-variable hybrid class for these types of polynomials is introduced and some of its properties are studied. For these hybrids polynomial, one can establish that there are recurrence relations, summation formulates, and symmetry identities. Thanks to the validity of these techniques, the hybrid polynomial families can be studied further.

Conclusion

In this article, we show Hermite polynomials and some of their generalizations. Likewise, Bernoulli polynomials, Euler polynomials, Genocchi polynomials and Frobenius-Euler polynomials are mentioned. Finally taking the definitions of these polynomials, we study the generating functions of the new families of Hermite type polynomials such as the following, the Hermite-Bernoulli polynomials, the Hermite-Euler polynomials, the Hermite-Genocchi polynomials and the Hermite-Frobenius-Euler polynomials.

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