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Research Article

Problem of Stability of Solutions of the Generalized Nonlinear Schrödinger Equation in Nonuniform and Nonstationary Media

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Abstract

On the basis of the analytical and numerical approaches the stability and dynamics of interaction of the multidimensional soliton-like solutions of the generalized nonlinear Schrödinger equation, which describes the waves in a plasma, fiber and planar optical waveguides, taking into account inhomogeneity and nonstationarity of propagation medium, is studied. The sufficient conditions of stability of the 2-dimensional and 3-dimensional solutions are obtained, and it is shown that even in the simplest 1-dimensional case the GNLS equation can have stable and quasi-stable solutions of the soliton and breather types and also unstable solutions which disperses with time. Obtained results can be useful in numerous applications in plasma physics, nonlinear optics and in many other fields of physics.

Keywords: Generalized Nonlinear Schrödinger Equation; Envelop Solitons; Breathers; Interaction; Multidimensional Solitons; Nonuniform Medium; Nonstationary Medium; Plasma; Fiber; Optical Waveguide

Introduction

If in the BK[1] system [1,2]:

$$\partial_t u + \hat{A}(t,u)u = f, \quad f = \sigma\int_{-\infty}^{x}\Delta_\perp u \mathrm{d}x + f', \quad \Delta_\perp = \partial_y^2 + \partial_z^2 \qquad (1)$$

operator has form $\hat{A}(t,u) = i[\gamma|u|^2 - \beta\partial_x^2] + \alpha/2$, that it is the 3-dimensional (3D) generalized nonlinear Schrödinger equation (3-GNLS) [3]:

$$\partial_t u + i\gamma|u|^2 u - i\beta\partial_x^2 u + (\alpha/2)u = \sigma\int_{-\infty}^{x}\Delta_\perp u \,\mathrm{d}x + f', \qquad (2)$$

where $\alpha, \beta, \gamma = \varphi(t,x,y,z)$, $f' = f'(t,x,y,z)$, and $(\alpha/2)u$ describes dissipative effects, and u is an envelope of the wave packet (pulse). Equation 3-GNLS (2) describes the dynamics of the envelope of modulated nonlinear waves and pulses (wave packets) in dispersive media and has many important applications in plasma physics (for example, it describes the propagation of the Langmuir waves in a hot plasma), nonlinear optics (propagation of light pulses in crystals, optical fibers and flat optical waveguides), it describes, in particular, such phenomena as turbulence, wave collapse and optical self-focusing. Equation (2) is also used in other areas of physics, such as, for example, the theory of superconductivity and low-temperature physics (in particular, the usual NLS equation is a simplified 1D form of the Ginzburg-Landau equation [4], first introduced by them in 1950 when describing superconductivity), low-amplitude gravitational waves on the surface of a deep inviscid liquid, etc. Note, that 3D equation (2) is not completely integrable, and its analytical solutions are unknown in general case (except, perhaps, for smooth solutions like solitary waves). But, using the approaches developed in [5, 6] for other equations of the BK system [the

[1]Belashov-Karpman (BK) system

GKP equation, when in (1) $\hat{A}(t,u) = \alpha u \partial_x - \partial_x^2(\nu - \beta\partial_x - \gamma\partial_x^3)$ and the 3-DNLS equation, in case if $\hat{A}(t,u) = 3s\,|\,p\,|^2\,u^2\partial_x - \partial_x^2(i\lambda + \nu)$], we can investigate the stability of possible solutions to the 3-GNLS equation, that is the purpose of this paper.

Analysis of the solutions' stability

Let us write (2) with α = 0 (3-NLS equation) in Hamiltonian form:

$$\partial_t u = \partial_x(\delta H/\delta u), \quad (3)$$

where

$$H = \int_{-\infty}^{\infty}\left[\tfrac{\gamma}{2}|\,u\,|^4 + \beta u u^*\partial_x\varphi + \tfrac{1}{2}\sigma(\nabla_\perp\partial_x w)^2\right]d\mathbf{r}, \quad \partial_x^2 w = u, \quad \varphi = \arg(u)$$

Using the method detailed in [1, 5, 6], we investigate the stability of 2D and 3D solutions of equation (2). At this, the problem for equation (3) is formulated in the form of the variational equation $\delta(H + \upsilon P_x) = 0,\ P_x = \frac{1}{2}\int u^2 d\mathbf{r}$, the meaning of which is that all finite solutions of equation (3) are stationary points of the Hamiltonian H at a fixed value of the momentum projection P_x. In accordance with Lyapunov's stability theorem, in a dynamical system, the points that correspond to the minimum or maximum of the Hamiltonian H are absolutely stable. If the extremum is local, locally stable solutions will correspond to it.

Let us consider the deformations of H conserving the momentum projection P_x:

$$u(x, r_\perp) \to \zeta^{-1/2}\eta^{-1}u(x/\zeta,\ \mathbf{r}_\perp/\eta), \qquad \zeta, \eta \in C.$$

The Hamiltonian takes form $H(\zeta,\eta) = a\zeta^{-1}\eta^{-2} + b\zeta^{-1} - c\zeta^2\eta^{-2}$ with the coefficients

$$a = (\gamma/2)\int|u|^4 d\mathbf{r},\quad b = \beta\int u u^*\partial_x\varphi\, d\mathbf{r},\quad c = (\sigma/2)\int(\nabla_\perp\partial_x w)^2 d\mathbf{r}.$$

From the necessary conditions of the extremum, $\partial_\zeta H = 0,\ \partial_\eta H = 0$, we find, at once, its coordinates:

$$\zeta_0 = -ac^{-1}, \quad \eta_0 = \left[-ab^{-1}\left(1 + a^2c^{-2}\right)\right]^{1/2},$$

where $b < 0$ if $\eta \in R \subset C$ since $a > 0$, $c > 0$ by definition, and $b > 0$ if $\eta \in C$. Sufficient conditions of a minimum at point (ζ_i, η_j) are the following:

$$\begin{vmatrix} \partial_\zeta^2 H(\zeta_i,\eta_j) & \partial_{\zeta\eta}^2 H(\zeta_i,\eta_j) \\ \partial_{\eta\zeta}^2 H(\zeta_i,\eta_j) & \partial_\eta^2 H(\zeta_i,\eta_j) \end{vmatrix} > 0, \quad \partial_\zeta^2 H(\zeta_i,\eta_j) > 0.$$

Solving this set of the inequalities we obtain that for the waves in case $b < 0$ (positive nonlinearity) $a/c < d = (2\sqrt{2})^{-1}\sqrt{13 + \sqrt{185}}$, whence it follows that $H > -3bd/(1 + 2d^2)$, that is the Hamiltonian is bounded from below. At $b > 0$ (negative nonlinearity): the change $b \to -b$ is equivalent to change $y \to -iy$, $z \to -iz$ and $H < -3bd/(1 + 2d^2)$, that is the Hamiltonian is not bounded from below (it is bounded from above).

So, we have proved the possibility of the existence of stable 3D solutions in the 3-NLS model and obtained the conditions of their stability, that is, we have found the ranges of values of the coefficients of the equation (the variable in time and space characteristics of the medium) when the 3D solitons are stable.

Confirmation of the analytical results by numerical simulation

The results of numerical modeling of the 3-GNLS equation for the general case of an inhomogeneous and nonstationary medium confirm the conclusions made on the basis of an analytical consideration of the problem. As an illustration, Figures 1 and 2 show the results obtained for σ = 0 (1D case) and initial conditions in form of the soliton-like envelope pulse:

$$u(x,0) = A\exp(-x^2/l)$$

and

$$u(x,0) = A\exp[-(x-5)^2/l] + A\exp[-(x+5)^2/l],$$

respectively, in the simplest case of the NLS equation with β, γ = const (stationary medium); α, f' = 0 at negative nonlinearity, β > 0. In this case $b > 0$ and the Hamiltonian $H > -3bd/(1+2d^2)$, and therefore the stability condition for negative nonlinearity, $H < -3bd/(1+2d^2)$, is not satisfied, and, as can be seen from the figures, we observe the scattering of the envelop pulses with time.

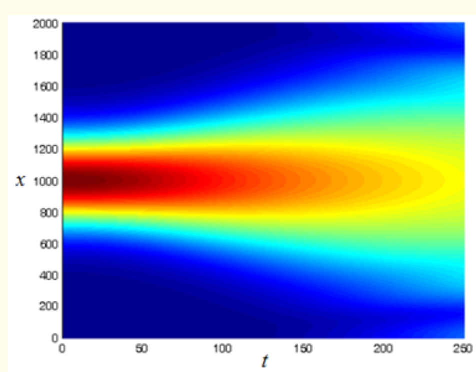

Figure 1: Evolution of the Gaussian envelop pulse at $A = 2$, $l = 2$; $\beta = 0.5$, $\gamma = 0$.

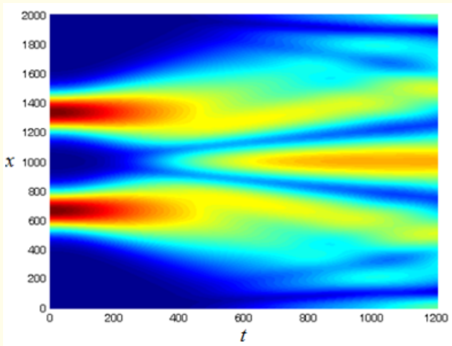

Figure 2: Evolution of the Gaussian 2-pulse envelop perturbation at при $A = 1$, $l = 4$; $\beta = 0.5$, $\gamma = 0$.

Figure 3 shows two examples of the results of evolution of the Gaussian pulse in a nonstationary medium with negative nonlinearity when the stability condition H < $-3bd/(1+2d^2)$ is satisfied. As a result of evolution, in this case, the appearance of powerful stable pulsations of the breather type from the initial solitary pulse is observed.

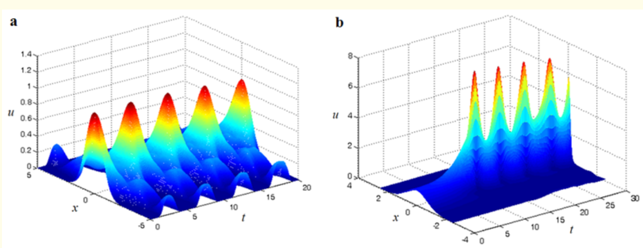

Figure 3: Evolution of the Gaussian envelop pulse in nonstationary medium at $\alpha, f' = 0$: a) $\beta = 0.5$, $\gamma = -1+0.01\sin 2\pi t$; b) $\gamma = -1$, $\beta(t) = -0.5$ for $T \le 5$ and $\beta(t) = 0.5(1+0.2\sin 2\pi t)$ for $t > 5$; the cases of negative nonlinearity.

The examples of the interaction of the soliton-like initial pulses of form

$$u(x,0) = A[\text{sch}(x) + \text{sch}(x - s/2) + \text{sch}(x + s/2)], \quad u(x,0) = A[\text{sch}(x - s/2) + \text{sch}(x + s/2)] \tag{4}$$

at negative nonlinearity in the GNLS model are shown in figures 4 and 5, respectively. In the first case, the stability condition is not satisfied, and at the first stage we observe the appearance of one powerful pulse from a 3-pulse initial perturbation and then, with time, its decay into two pulses of small amplitude. In the second case, the stability condition is satisfied, and a stable evolution of the 2-pulse perturbation takes place. In numerical experiments, it was also found that for weak negative nonlinearity, when the stability condition is satisfied, the transition from stable evolution to the regime of stable pulsations (breathers) occurs when the initial distance *s* in (4) between pulses decreases.

A detailed numerical study of the problems of evolution and interaction of 2D and 3D pulses in the 3GNLS model was presented in [3, 7-9].

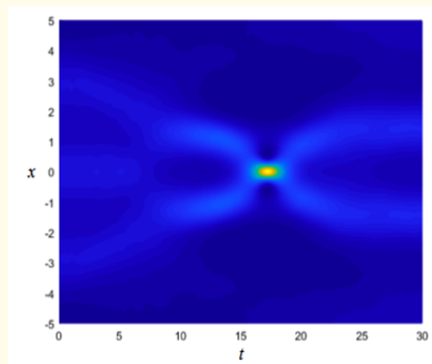

Figure 4: Interaction of the three GNLS pulses (stationary medium) at $\gamma = -1$, $\beta = 0.25$; a case of weak negative nonlinearity.

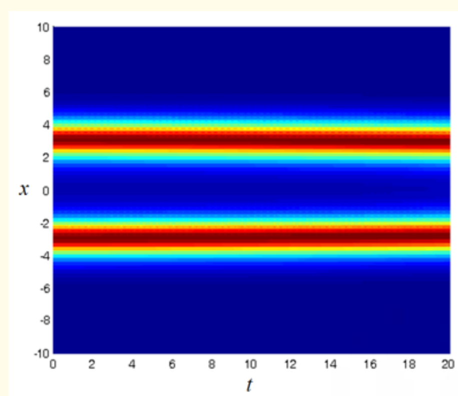

Figure 5: Lack of the GNLS impulses' interaction (stationary medium) at $\gamma = -1$, $\beta = 0.05$; a case of negative nonlinearity.

Conclusion

Summarizing the results, we note that we have analytically obtained the stability conditions for soliton-like solutions of the GNLS equation, which are confirmed by numerical study of cases of stable and unstable (with the formation of breathers) evolution of pulses of various shapes, as well as the interaction of 2- and 3-pulse structures, leading to the formation of stable and unstable solutions.

Acknowledgments

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